

# A new rational conformal field theory extension of the fully degenerate $W_{1+\infty}^{(m)}$

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**ABSTRACT:** We found new identities among the Dedekind  $\eta$ -function, the characters of the  $W_m$  algebra and those of the level 1 affine Lie algebra  $\widehat{su(m)}_1$ . They allow to characterize the  $\mathbb{Z}_m$ -orbifold of the  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  (our theory TM) as an extension of the fully degenerate representations of  $W_{1+\infty}^{(m)}$ . In particular, TM is proven to be a  $\Gamma_\theta$ -RCFT extension of the chiral fully degenerate  $W_{1+\infty}^{(m)}$ .

**KEYWORDS:** Field Theories in Lower Dimensions, Conformal and W Symmetry.

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**1. Introduction**

In this paper, we show that our theory TM, introduced in [1] to describe a quantum Hall fluid at Jain fillings, gives a new rational conformal field theory (RCFT) extension of the  $W_{1+\infty}^{(m)}$  chiral algebra corresponding to the so called irreducible fully degenerate representations [2–4].

A chiral algebra  $\mathfrak{A}$  in a CFT is generated by the modes (the Fourier components) of the conserved currents; the Virasoro algebra [5, 6] is the chiral algebra corresponding to the analytic component  $T(z)$  of the stress-energy tensor. A CFT can be characterized by the corresponding chiral algebra and the set of its irreducible positive energy (highest weight) representations closed under the fusion algebra. A mathematical introduction to the subject of vertex or chiral algebras can be found in [7, 8] and references there in.

An extended chiral algebra  $\mathfrak{A}^{Ex.}$  is itself a chiral algebra obtained by adding to the original  $\mathfrak{A}$  the modes of further conserved currents. The highest weight (h.w.) representations of  $\mathfrak{A}^{Ex.}$  are opportune collections of h.w. representations of  $\mathfrak{A}$ , so that any h.w.  $\mathfrak{A}^{Ex.}$ -module is the direct sum of the corresponding collection of h.w.  $\mathfrak{A}$ -modules.

Let us remember that RCFTs are CFTs with a finite set of h.w. representations closed under the fusion algebra [9]. The representations of the Virasoro algebra with central charge  $c \geq 1$  are not RCFTs [10], and so RCFTs with  $c \geq 1$  correspond always to extensions of the Virasoro algebra. An RCFT can be defined [11] by reorganizing the set (possibly infinite)

of Virasoro h.w. representations of the CFT into a finite number of their collections closed under the fusion algebra, the last ones being the h.w. representations of the RCFT.

Let  $\mathcal{X}$  be the finite set parametrizing the h.w. representations of the RCFT, then the fusion algebra is defined on  $A(\mathcal{X}) := \bigoplus_{x \in \mathcal{X}} \mathbb{C}x$  by introducing the product of representations:

$$x \circ y := \sum_{z \in \mathcal{X}} N_{x,y,z} C z, \quad (1.1)$$

where  $N_{x,y,z}$  are called fusion coefficients and  $C$  is the finite dimensional matrix representing the charge conjugation. The Verlinde formula for an RCFT [9, 12] expresses the fusion coefficients  $N_{x,y,z}$  as a function of the elements of the symmetric unitary finite dimensional matrix  $S = \|S_{x,y}\|_{x,y \in \mathcal{X}}$ , representing the action of the modular transformation  $S \in PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\mathbb{Z}_2$  on the characters of the RCFT, or explicitly:

$$N_{x,y,z} = \sum_{a \in \mathcal{X}} S_{a,x} S_{a,y} S_{a,z} / S_{a,e}, \quad (1.2)$$

where  $e \in \mathcal{X}$  parametrizes the unique h.w. representation including the vacuum vector. Furthermore, the action of the charge conjugation on the h.w. representations is given by  $C = \|S_{x,y}\|_{x,y \in \mathcal{X}}^2$ . The fusion algebra  $(A(\mathcal{X}), \circ)$  is then a finite dimensional commutative associative semisimple algebra with unity  $e$ .

The Verlinde formula (1.2), in particular, makes possible to characterize an RCFT by its properties under modular transformations.

Let  $\tilde{\Gamma}$  be a subgroup of the modular group  $PSL(2, \mathbb{Z})$  containing  $S$ , then a CFT whose characters define a finite dimensional representation of  $\tilde{\Gamma}$  is an RCFT. In the following, we denote this kind of RCFT as a  $\tilde{\Gamma}$ -RCFT to underline the subgroup  $\tilde{\Gamma}$  of the modular group  $PSL(2, \mathbb{Z})$ .

These concepts are here applied to define a new RCFT extension of the CFT with chiral algebra  $W_{1+\infty}^{(m)}$  and with h.w. representations the irreducible fully degenerate ones [2–4]. From now on, we simply refer to such a CFT as to the fully degenerate  $W_{1+\infty}^{(m)}$ ; as it is well known, this CFT is not a rational one. Indeed, there are infinitely many irreducible fully degenerate representations and all are required to be closed under fusion, as the study of the corresponding characters and modular transformations shows [3].

In the literature there are many classes of RCFT extensions of the fully degenerate  $W_{1+\infty}^{(m)}$ . Examples are the affine level 1 chiral algebra  $\mathfrak{A}_1(u(m))$  or  $\mathfrak{A}_1(so(2m))$ , where  $W_{1+\infty}^{(m)}$  coincides with the corresponding  $U(m)$ -invariant subalgebra. More general RCFT extensions of the fully degenerate  $W_{1+\infty}^{(m)}$  are the lattice chiral algebras  $\mathfrak{A}(Q)$  associated with the compact group  $U(m)$ , where  $Q$  is a rank  $m$  integral lattice including as a sublattice the rank  $m - 1$   $su(m)$  lattice (see [8, section 5]). The  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  (described in section 4 of this paper) can be seen as a class of examples of these RCFT extensions.

The orbifold construction is a way to define new RCFTs starting from a given RCFT by quotienting it with a generic discrete symmetry group  $G$ . More precisely, let  $G$  be a discrete group of automorphisms of the chiral algebra  $\mathfrak{A}$  of the original RCFT, then the corresponding orbifold chiral algebra  $\mathfrak{A}^G := \mathfrak{A}/G$  is the subalgebra of  $\mathfrak{A}$  defined as the

invariant part of  $\mathfrak{A}$  under  $G$ . The  $G$ -orbifold RCFT with chiral algebra  $\mathfrak{A}^G$  has a finite set of irreducible representations that splits in two sectors. The untwisted sector of  $\mathfrak{A}^G$  has irreducible representations that coincide with those of the original chiral algebra  $\mathfrak{A}$  or with opportune restrictions of them. The twisted sector of  $\mathfrak{A}^G$  has instead irreducible representations that cannot be expressed in terms of those of  $\mathfrak{A}$ .

In [8], the orbifold construction is shown to be a tool to obtain a class of RCFT extensions of the fully degenerate  $W_{1+\infty}^{(m)}$ . The lattice chiral algebras<sup>1</sup>  $\mathfrak{A}(\mathbb{Z}^m)$  is one of the above RCFT extensions of the fully degenerate  $W_{1+\infty}^{(m)}$  with a unique irreducible representation [3]. To such an RCFT<sup>2</sup> is applied the orbifold construction with respect to  $G$ , a discrete group of inner automorphisms<sup>3</sup> of  $\mathfrak{A}(\mathbb{Z}^m)$ , obtaining a class of RCFT extensions of the fully degenerate  $W_{1+\infty}^{(m)}$ .

In this paper, we show that our theory TM [1], characterized as the cyclic permutation orbifold [13, 14] with respect to the outer automorphisms [15, 16]  $\mathbb{Z}_m$  of the chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$ , is a  $\Gamma_\theta$ -RCFT extension of the fully degenerate  $W_{1+\infty}^{(m)}$ .

The results given here contain, in particular, a generalization to any prime  $m$  of the  $m = 2$  special case presented in [1]. For  $m$  not a prime number the results still hold and will be the subject of a forthcoming paper.

The paper is organized as follows. In section 2, we review the CFT with chiral algebra  $W_{1+\infty}^{(m)}$  and the decomposition of the affine level 1  $\widehat{su(m)}_1$  characters in terms of those of the  $\mathcal{W}_m$  chiral algebra [17]. In section 3, we derive the main identities among the  $\eta$ -function of Dedekind, the characters of the  $\mathcal{W}_m$  chiral algebra and the characters of the affine level 1  $\widehat{su(m)}_1$ , evaluated at the so called principal element of type  $\rho$  of B. Kostant [18]. In section 4, we recall the definition of the  $\Gamma_\theta$ -RCFT  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  [19–22]. The corresponding chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  is identified together with the finite set of h.w. representations (modules). The corresponding characters and modular transformations are given. In section 5, we derive our theory TM by making the explicit  $\mathbb{Z}_m$  cyclic permutation orbifold construction of the  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ . In particular, a finite set of irreducible (h.w.) representations (modules) of the orbifold chiral algebra  $\mathfrak{A}_{\text{TM}} := \mathfrak{A}^{\mathbb{Z}_m}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  is found. By explicitly performing the modular transformations of the corresponding characters, we prove that they provide a unitary finite dimensional representation of the modular subgroup  $\Gamma_\theta$ , i.e. TM is a  $\Gamma_\theta$ -RCFT. In section 6, we show, using the identities derived in section 3, that TM gives a  $\Gamma_\theta$ -RCFT extension of the fully degenerate  $W_{1+\infty}^{(m)}$ . In section 7, some final remarks are contained. Finally, we report in two appendices some useful definitions and results. In appendix A, we recall the definition of the  $\Gamma_\theta$  subgroup of the modular group  $PSL(2, \mathbb{Z})$ . In appendix B, we recall the definition of the  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_q$  [22], where  $q$  is odd. The corresponding chiral algebra  $\mathfrak{A}(\widehat{u(1)}_q)$  is identified together with the finite set of the h.w. representations (modules). The corresponding characters and their modular transformations are also given.

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<sup>1</sup>Where  $\mathbb{Z}^m$  is the rank  $m$  orthonormal lattice.

<sup>2</sup>See [8, Theorem 5.2].

<sup>3</sup>In particular,  $G$  is a finite subgroup of  $U(m)$ .

## 2. The $W_{1+\infty}^{(m)}$ chiral algebra

$W_{1+\infty}$  is the unique nontrivial central extension [23, 24] of the Lie algebra  $w_\infty$  [25] of the area-preserving diffeomorphisms on the circle; its representation theory was developed in [2–4].  $W_{1+\infty}$  has an infinite number of generators  $W_m^\nu$ , with  $\nu$  a non negative integer and  $m \in \mathbb{Z}$ , satisfying the commutation relations:

$$\left[ W_n^\nu, W_{n'}^{\nu'} \right] = (\nu'n - \nu n') W_{n+n'}^{\nu+\nu'-1} + \dots + c \frac{(\nu!)^4}{(2\nu)!} \binom{n+\nu}{n-\nu-1} \delta_{\nu,\nu'} \delta_{n+n',0}, \quad (2.1)$$

where dots denote a finite number of similar terms involving the operators  $W_{n+n'}^{\nu+\nu'-1-2l}$ . The generators  $W_n^\nu$  of  $W_{1+\infty}$  define the modes of a Heisenberg algebra  $\widehat{u(1)}$ , for  $\nu = 0$ , and those of a Virasoro algebra, for  $\nu = 1$ , with central charge  $c$ . The unitary representations of  $W_{1+\infty}$  have positive integer central charge  $c = m \in \mathbb{N}$  and their h.w. representations are defined by the h.w. vectors  $|\mathbf{r}\rangle$ , where  $\mathbf{r} := (r_1, \dots, r_m)$  is an  $m$ -dimensional vector with real values. The h.w. vector  $|\mathbf{r}\rangle$  is defined by:

$$W_0^\nu |\mathbf{r}\rangle = w_\nu(\mathbf{r}) |\mathbf{r}\rangle \quad \text{for } \nu \geq 0, \quad W_m^\nu |\mathbf{r}\rangle = 0 \quad \text{for } \nu \geq 0 \quad m > 0, \quad (2.2)$$

with eigenvalues [8]:

$$w_\nu(\mathbf{r}) = \frac{(\nu-1)! \nu!}{(2\nu)!} \sum_{j=0}^{\nu-1} \binom{\nu}{j} \binom{\nu}{j+1} \sum_{i=1}^m r_i (r_i - j) \dots (r_i + \nu - j - 1). \quad (2.3)$$

Thus, in particular,  $|\mathbf{r}\rangle$  is a h.w. vector for the Virasoro algebra defined by  $L_m := W_m^1$ ,  $m \in \mathbb{Z}$ , with conformal dimension  $h_{\mathbf{r}} := w_1(\mathbf{r}) = (1/2) \sum_{i=1}^m r_i^2$ . The unitary irreducible (h.w.) module  $\mathbb{W}_{\mathbf{r}}^{(m)}$  of  $W_{1+\infty}^{(m)}$  with central charge  $c = m$  is built by the action of the generators  $W_n^\nu$  on the h.w. vector  $|\mathbf{r}\rangle$  quotient the submodule generated by the unique singular vector of degree  $m+1$  [2]. The unitary irreducible h.w. representations of  $W_{1+\infty}^{(m)}$  are given in terms of those of the  $m$ -component free bosons  $\widehat{u(1)}^{\otimes m}$ . They can be of two types: generic or degenerate. The h.w. representations defined by  $|\mathbf{r}\rangle$  are generic if  $\mathbf{r} = (r_1, \dots, r_m)$  satisfies the conditions  $r_a - r_b \notin \mathbb{Z}$ ,  $\forall a \neq b \in \{1, \dots, m\}$ , while they are degenerate if  $r_a - r_b \in \mathbb{Z}$ , for some  $a \neq b$ . Finally, the h.w. representations are fully degenerate if  $\mathbf{r} = (r_1, \dots, r_m)$  satisfies the conditions  $r_a - r_b \in \mathbb{Z}$ ,  $\forall a \neq b \in \{1, \dots, m\}$ .

The irreducible fully degenerate representations of  $W_{1+\infty}^{(m)}$  are isomorphic [22] to those of  $\widehat{u(1)} \otimes \mathcal{W}_m$ , where  $\mathcal{W}_m$  is the algebra with central charge  $c = (m-1)$  defined by the limit  $a \rightarrow \infty$  of the Zamolodchikov-Fateev-Lukyanov algebra with  $c = (m-1) \left( 1 - \frac{m(m+1)}{a(a+1)} \right)$  [26]. The irreducible fully degenerate representations of  $W_{1+\infty}^{(m)}$  are classified with h.w.  $\mathbf{r}$  satisfying the additional condition that its elements are arranged in a decreasing order, that is  $\mathbf{r} \in \mathbb{P}^{(m)}$ , where  $\mathbb{P}^{(m)} := \{ \mathbf{r} \in \mathbb{R}^m : r_1 \geq \dots \geq r_m, r_a - r_b \in \mathbb{Z}, \forall a \neq b \in \{1, \dots, m\} \}$ . It is worth noticing that for any h.w.  $\mathbf{r} \in \mathbb{P}^{(m)}$ , defining an irreducible fully degenerate represen-

tation of  $W_{1+\infty}^{(m)}$ , a h.w.  $\Lambda$  of  $su(m)$  is defined in the following way:

$$\Lambda := \sum_{i=1}^{m-1} \lambda_i \Lambda_i, \quad \lambda_i := r_i - r_{i+1} \in \mathbb{Z}_+, \quad (2.4)$$

where  $\Lambda_i$  are the fundamental weights of  $su(m)$  and  $\lambda_i$  are the Dynkin labels.

The  $\mathcal{W}_m$  chiral algebra can be also defined by a coset construction of the kind  $W[\hat{\mathfrak{g}}_k/\mathfrak{g}; k]$  based on the Casimir operators of a finite algebra  $\mathfrak{g}$  (see [17] for details). In particular, the coset that defines  $\mathcal{W}_m$  is  $W[\widehat{su(m)}_k/su(m); k=1]$  and involves the finite algebra  $su(m)$ ; thus, the central charge of the CFT with chiral algebra  $\mathcal{W}_m$  has the same value of that of the level 1 affine  $\widehat{su(m)}_1$ , i.e.  $c_{\mathcal{W}_m} = c_{\widehat{su(m)}_1} = m - 1$ .

In the following, we will make use of characters for clarifying the relations among the representations of the chiral algebras under study. Indeed, to any h.w. representation of a chiral algebra we can associate a character which accounts for the main properties of the representation. The explicit form of the character depends on the nature of the *chiral algebra*. In the particular case of a Lie algebra or of a Kac-Moody algebra (see chapter 9 of [27]) we can define the *formal character*, corresponding to a given h.w.  $\Lambda$ , as the *formal function*:

$$\chi_{\Lambda}^{\mathfrak{g}} := \sum_{\Lambda' \in \Omega_{\Lambda}} \text{mult}_{\Lambda}(\Lambda') e^{2\pi i \Lambda'}, \quad (2.5)$$

where  $\Omega_{\Lambda}$  is the set of the weights in the h.w.  $\Lambda$  representation of the algebra  $\mathfrak{g}$ ,  $\text{mult}_{\Lambda}(\Lambda')$  is the multiplicity<sup>4</sup> of the weight  $\Lambda'$  and  $e^{\Lambda'}$  denotes a formal exponential satisfying:

$$e^{\Gamma_1} e^{\Gamma_2} = e^{\Gamma_1 + \Gamma_2} \quad \text{and} \quad e^{\Gamma}(\xi) = e^{(\Gamma, \xi)}, \quad (2.6)$$

where  $(,)$  is the bilinear form (Killing form) on  $\mathfrak{g}$  and  $\xi$  is an arbitrary element of the dual Cartan subalgebra. The action of the exponential  $e^{\Gamma}$  on  $\xi$  allows to compute  $\chi_{\Lambda}^{\mathfrak{g}}(\xi)$ , the so called *specialization* of the character at  $\xi$ . In the case of Lie algebras, the group character definition, given in the representation theory of Lie groups (see [28]), is simply related to that of the formal character (as explained in [29, section 13.4.1]), so from now on we will refer to it as the character associated to a h.w. representation of the algebra. For more general chiral algebras the definition of character can be given analogously. In this paper, in particular, we will define and study the characters for a class of chiral algebras which are cyclic orbifold of lattice chiral algebras (see [7, 8] for a general definition of the characters of lattice chiral algebras).

Let us now consider the characters of the coset  $\mathcal{W}_m$ ; by definition, the h.w. representations of  $\mathcal{W}_m$  are defined by decomposing those of  $\widehat{su(m)}_1$  in terms of those of  $su(m)$ . Thus, the characters of the coset  $\mathcal{W}_m$  are the *branching functions* constructed by decomposing

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<sup>4</sup>That is,  $\text{mult}_{\Lambda}(\Lambda')$  is the dimension of the eigenspace  $V_{\Lambda}^{\Lambda'}$  with eigenvalue  $\Lambda'$  in the weight space decomposition of the h.w. module  $V_{\Lambda}$ :

$$V_{\Lambda} = \sum_{\Lambda' \in \Omega_{\Lambda}} V_{\Lambda}^{\Lambda'}.$$

the characters of  $\widehat{su(m)}_1$  in terms of those of  $su(m)$ :

$$\chi_{\widehat{\Lambda}_l}^{su(m)_1}(\xi|\tau) = \sum_{\Lambda \in P_+ \cap \Omega_l} b_{\widehat{\Lambda}_l}^\Lambda(\tau) \chi_\Lambda^{su(m)}(\xi), \quad (2.7)$$

where  $\Omega_l$  is the finite part of  $\widehat{\Omega}_l$ , the set of the affine weights in the integrable h.w. representation of  $\widehat{su(m)}_1$  corresponding to the fundamental weight  $\widehat{\Lambda}_l$ , and  $P_+$  is the set of the dominant weights of the Lie algebra  $su(m)$ . Therefore, the character of  $\mathcal{W}_m$  corresponding to the h.w.  $\Lambda$  is defined by  $\chi_\Lambda^{\mathcal{W}_m}(\tau) := b_{\widehat{\Lambda}_l}^\Lambda(\tau)$  and explicitly given by:

$$\chi_\Lambda^{\mathcal{W}_m}(\tau) = \frac{q^{\frac{\Lambda^2}{2}}}{\eta(\tau)^{m-1}} \prod_{\alpha \in \Delta_+} (1 - q^{(\Lambda+\rho, \alpha)}), \quad (2.8)$$

where  $q := e^{2\pi i \tau}$ ,  $\rho := \left(\sum_{\alpha \in \Delta_+} \alpha\right)/2$  is the Weyl vector,  $\Delta_+$  is the set of the positive roots of  $su(m)$ ,  $\Lambda = \Lambda_l + \gamma$  with  $\Lambda_l$  a fundamental weight of  $su(m)$  and  $\gamma \in Q$ , the set of roots of  $su(m)$ . While, according to the *Weyl character formula*, the character corresponding to the h.w.  $\Lambda$  of  $su(m)$  is:

$$\chi_\Lambda^{su(m)}(\xi) = \frac{\sum_{w \in W} \epsilon(w) e^{2\pi i (w(\Lambda+\rho), \xi)}}{\sum_{w \in W} \epsilon(w) e^{2\pi i (w\rho, \xi)}}, \quad (2.9)$$

where  $w$  is an element in the Weyl group  $W$  of  $su(m)$ ,  $\epsilon(w) = (-1)^{l(w)}$  is the signature of the Weyl reflection  $w$  and  $l(w) \in N$  is the length of  $w$ .

For  $z \rightarrow 0$  the character  $\chi_\Lambda^{su(m)}(\xi = z\rho)$  goes to the dimension of the h.w.  $\Lambda$  representation of the finite algebra  $su(m)$ ,  $d_{su(m)}(\Lambda)$ . Thus, the characters of  $\widehat{su(m)}_1$ , specialized to the weight  $\hat{\xi} = \left(z\rho + \tau\widehat{\Lambda}_0\right)\Big|_{z=0}$ , can be written in the form:

$$\chi_l^{\widehat{su(m)}_1}(\tau) := \lim_{z \rightarrow 0} \chi_{\widehat{\Lambda}_l}^{su(m)_1}(\xi = z\rho|\tau) = \sum_{\Lambda \in P_+ \cap \Omega_l} d_{su(m)}(\Lambda) \chi_\Lambda^{\mathcal{W}_m}(\tau). \quad (2.10)$$

The above analysis on the h.w.  $\Lambda$  representations of  $\mathcal{W}_m$  and the characterization given of the irreducible fully degenerate h.w.  $\mathbf{r} \in \mathbb{P}^{(m)}$  of  $W_{1+\infty}^{(m)}$  imply the following expression for the corresponding character:

$$\chi_{\mathbf{r}}^{\mathcal{W}_m}(w|\tau) := Tr_{\mathbb{W}_{\mathbf{r}}^{(m)}} \left( e^{2\pi i \tau (L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \chi_\Lambda^{\mathcal{W}_m}(\tau) \frac{e^{2\pi i \left\{ \tau \frac{1}{2m} (\sum_{i=1}^m r_i)^2 + w \mathbf{r} \mathbf{t} \det(\mathbf{R}) \right\}}}{\eta(\tau)}, \quad (2.11)$$

where  $\mathbf{t}^{\mathbf{T}} := (1, \dots, 1)$ ,  $L_0$  is the zero mode of the Virasoro algebra,  $J$  is the conformal charge defined by  $J := \mathbf{r} \mathbf{t} \det(\mathbf{R})$ , and  $\mathbf{R}$  is the  $m \times m$  symmetric positive definite compactification matrix of the system of  $m$  free boson fields (see section 4 for details).

The h.w.  $\Lambda$  of  $su(m)$  in (2.11) is defined by (2.4) in terms of the h.w.  $\mathbf{r} \in \mathbb{P}^{(m)}$  and the identity (2.11) follows as a consequence of the following relation between the conformal dimensions  $h_{\mathbf{r}}$  and  $h_\Lambda$ :

$$h_{\mathbf{r}} = \frac{1}{2m} \left( \sum_{i=1}^m r_i \right)^2 + h_\Lambda, \quad (2.12)$$

where:

$$h_{\mathbf{r}} := \frac{1}{2} \sum_{i=1}^m r_i^2 \quad \text{and} \quad h_{\Lambda} := \frac{|\Lambda|^2}{2} = \frac{1}{2} \sum_{i=1}^m \left( r_i - \frac{1}{m} \sum_{j=1}^m r_j \right)^2. \quad (2.13)$$

### 3. The main identities

In this section, we derive the main results of the paper that allow us to show that our theory TM gives a new  $\Gamma_{\theta}$ -RCFT extension of the fully degenerate  $W_{1+\infty}^{(m)}$ .

**Proposition 1.** *Let  $\chi_{\widehat{\Lambda}_l}^{\widehat{su(m)}_1}(\xi|\tau)$  be the character of the affine level 1  $\widehat{su(m)}_1$  corresponding to the fundamental representation  $\widehat{\Lambda}_l$ , then it has the following series expansion in terms of the characters of  $\mathcal{W}_m$ :*

$$\chi_{\widehat{\Lambda}_l}^{\widehat{su(m)}_1} \left( \xi = \frac{\rho}{m} | \tau \right) = \delta_{l,0} \sum_{\Lambda \in D_m} \epsilon(w_{\Lambda}) \chi_{\Lambda}^{\mathcal{W}_m}(\tau), \quad (3.1)$$

where  $D_m := \{ \Lambda \in P_+ : \exists! w_{\Lambda} \in W \rightarrow w_{\Lambda}(\Lambda + \rho) - \rho \in mQ \}$ ,  $\epsilon(w_{\Lambda})$  is the signature of the Weyl reflection  $w_{\Lambda}$ ,  $P_+$  is the set of the dominant weights and  $Q$  is the root lattice of the finite algebra  $su(m)$ .

Let  $\eta(\tau)$  be the Dedekind function, then it holds:

$$\chi_{\widehat{\Lambda}_l}^{\widehat{su(m)}_1} \left( \xi = \frac{\rho}{m} | \tau \right) = \delta_{l,0} \frac{\eta(\tau)}{\eta(m\tau)}. \quad (3.2)$$

To prove this Proposition we make use of some important results of B. Kostant already given in [18]. Here, we just recall those results which turn out to be useful for us and we restate them according to our notations.

**Kostant's Proposition 1.** *Let  $g$  be a simple Lie algebra. Then, either*

- (1)  $\forall w \in W \rightarrow w(\Lambda + \rho) - \rho \notin hQ$ , or
- (2)  $\exists! w_{\Lambda} \in W \rightarrow w_{\Lambda}(\Lambda + \rho) - \rho \in hQ$ , where  $h$  is the Coxeter number of  $g$ .

This Proposition, in particular, makes clear the definition given above of the set  $D_m$ , if one recalls that  $m$  is the Coxeter number of  $su(m)$ . Kostant defines the so called principal element of type  $\rho$  and here we give a definition of it in a way to be independent of the normalization of the Killing form.

**Definition 1.** *Let  $g$  be a simply laced Lie algebra and  $x_p$  be defined as an element of  $g$  such that  $(x_p, \alpha_i) := 1/h$ ,  $i \in \{1, \dots, h-1\}$ , where  $\alpha_i$  are the simple roots,  $(,)$  is the Killing form and  $h$  is the Coxeter number of  $g$ .*

*Then, every element of  $g$  conjugate, with respect to the Weyl group, to the element  $x_p$  is called principal of type  $\rho$ .*



The normalization of the Killing form that we chose is  $(\alpha_i, \alpha_i) = 2, i \in \{1, \dots, h-1\}$ , in this case  $(\rho, \alpha_i) = 1$  and thus  $x_p = \rho/h$ .

**Kostant's Proposition 2.** For any  $\Lambda \in P_+$  and  $\xi$  principal of type  $\rho$  it results  $\chi_\Lambda^g(\xi) \in \{-1, 0, 1\}$ , where  $\chi_\Lambda^g$  is the character of the representation  $\Lambda$ . In particular, it holds:

$$\chi_\Lambda^g(\xi) = \begin{cases} 0 & \text{for } \Lambda \notin D_h \\ \epsilon(w_\Lambda) & \text{for } \Lambda \in D_h \end{cases}, \tag{3.3}$$

where  $D_h := \{\Lambda \in P_+ : \exists! w_\Lambda \in W \rightarrow w_\Lambda(\Lambda + \rho) - \rho \in hQ\}$ .

In the case of  $su(m)$ , then  $x_p = \rho/m$  is a principal element of type  $\rho$  and the following identities hold:

$$\chi_\Lambda^{su(m)}\left(\xi = \frac{\rho}{m}\right) = \begin{cases} 0 & \forall \Lambda \notin D_m \\ \epsilon(w_\Lambda) & \forall \Lambda \in D_m \end{cases}. \tag{3.4}$$

We give now a first characterization of the weights of  $D_m$  in terms of the finite part of the affine weights of the fundamental representations.

The finite part<sup>5</sup>  $\Omega_l$  of the affine weight systems  $\widehat{\Omega}_l, l \in \{0, \dots, m-1\}$ , can be characterized by the so called  $m$ -ality, or congruence class, of the weights.

**Lemma 1.** The  $m$ -ality of the weight  $\Lambda = [\lambda_1, \dots, \lambda_{m-1}]$ , where  $\lambda_i$  are the Dynkin labels, is defined as  $k(\Lambda) = \frac{1}{m} \sum_{i=1}^{m-1} i\lambda_i \pmod{1}$ . Then, all the weights in  $\Omega_l$  have the same  $m$ -ality  $l/m, l \in \{0, \dots, m-1\}$ .

*Proof.* We have to prove that  $k(\Lambda) = l/m \forall \Lambda \in \Omega_l$ . This is of course true for the finite fundamental weights  $\Lambda_l$ , that is  $k(\Lambda_l) = l/m$ . Now, the generic weight  $\Lambda \in \Omega_l$  has the form  $\Lambda = \Lambda_l + \sum_{i=1}^{m-1} n_i \alpha_i$ , and by using  $\alpha_i = 2\Lambda_i - (1 - \delta_{i,1})\Lambda_{i-1} - (1 - \delta_{i,m-1})\Lambda_{i+1}$  we get  $k(\alpha_i) = \frac{(i+1)}{m} \delta_{i,m-1} = 0 \pmod{1}$ . Thus, the equalities  $k(\Lambda) = k(\Lambda_l) = l/m$  hold.  $\square$

**Lemma 2.** All the weights of  $D_m$  have zero  $m$ -ality, that is:  $D_m \subset P_+ \cap \Omega_0$ .

*Proof.* If  $\Lambda \in D_m$  then  $w_\Lambda(\Lambda + \rho) - \rho \in mQ$ , by definition for each element of the Weyl group,  $\exists! n_i \in \mathbb{Z} : w_\Lambda(\Lambda + \rho) = \Lambda + \rho + \sum_{i=1}^{m-1} n_i \alpha_i$ . The condition  $\Lambda \in D_m$  so implies that  $\exists! q_i \in m\mathbb{Z} : \Lambda + \sum_{i=1}^{m-1} n_i \alpha_i = \sum_{i=1}^{m-1} q_i \alpha_i$ , that is  $\exists! p_i \in \mathbb{Z} : \Lambda = \sum_{i=1}^{m-1} p_i \alpha_i$ . Thus,  $\Lambda$  has zero  $m$ -ality and  $\Lambda \in P_+ \cap \Omega_0$ .  $\square$

Thus, we can give the following proof of the identity (3.1) — The identity (3.1) is an immediate consequence of Lemma 2, (3.4) and (2.7) evaluated at  $\xi = \rho/m$ .  $\square$

We are now ready to prove the identity (3.2). Let us start by giving the following Lemma.

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<sup>5</sup>The  $m$ -ality of an affine weight  $\widehat{\Lambda} = [\lambda_0, \lambda_1, \dots, \lambda_{m-1}]$  is defined as the  $m$ -ality of its finite part  $\Lambda = [\lambda_1, \dots, \lambda_{m-1}]$ . So, lemma 1 implies that the affine weight system  $\widehat{\Omega}_l$  has the same  $m$ -ality of the finite part  $\Omega_l$ , i.e.  $l/m$ .

**Lemma 3.** *The ratio  $\eta(\tau)/\eta(m\tau)$  has the equivalent expression in terms of  $\Theta$ -functions:*

$$\frac{\eta(\tau)}{\eta(m\tau)} = \frac{1}{\eta(\tau)^{m-1}} \frac{\prod_{j=1}^m \Theta_3\left(w - \frac{j}{m}|\tau\right)}{\Theta_3\left(mw - \frac{m+1}{2}|m\tau\right)}. \quad (3.5)$$

*Proof.* The identity (3.5) is equivalent to the following one:

$$\frac{\Theta_3\left(mw - \frac{m+1}{2}|m\tau\right)}{\eta(m\tau)} = \prod_{j=1}^m \frac{\Theta_3\left(w - \frac{j}{m}|\tau\right)}{\eta(\tau)}. \quad (3.6)$$

Using the definition of  $\eta(\tau)$  and the expansion of  $\Theta_3(w|\tau)$  in terms of infinite products:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \Theta_3(w|\tau) = q^{-1/24} \eta(\tau) \prod_{n=1}^{\infty} (1 - yq^{n-1/2})(1 - y^{-1}q^{n-1/2}), \quad (3.7)$$

where  $q := e^{2\pi i\tau}$  and  $y := e^{2\pi iw}$ , we obtain:

$$\frac{\Theta_3\left(mw - \frac{m+1}{2}|m\tau\right)}{\eta(m\tau)} = q^{-m/24} \prod_{n=1}^{\infty} (1 - (-1)^{\frac{m+1}{2}} y^m q^{m(n-1/2)})(1 - (-1)^{\frac{m+1}{2}} y^{-m} q^{m(n-1/2)}) \quad (3.8)$$

and

$$\prod_{j=1}^m \frac{\Theta_3\left(w - \frac{j}{m}|\tau\right)}{\eta(\tau)} = q^{-m/24} \prod_{j=1}^m \prod_{n=1}^{\infty} (1 - e^{-\frac{2\pi i}{m}j} y q^{n-1/2})(1 - e^{-\frac{2\pi i}{m}j} y^{-1} q^{n-1/2}). \quad (3.9)$$

The identity in (3.6) is then an immediate consequence of the following one:

$$\prod_{j=1}^m \left(1 + a e^{-\frac{2\pi i}{m}j}\right) = \left(1 + (-1)^{\frac{m+1}{2}} a^m\right), \quad (3.10)$$

due to the properties of the roots of the unity. □

Furthermore, by using the series expansion of  $\Theta_3(w|\tau)$ , it holds:

$$\prod_{j=1}^m \Theta_3\left(w - \frac{j}{m}|\tau\right) = \sum_{(n_1, \dots, n_m) \in \mathbb{Z}^m} q^{\frac{1}{2} \sum_{i=1}^m n_i^2} e^{2i\pi \sum_{j=1}^m (w - \frac{j}{m}) n_j}. \quad (3.11)$$

If we define  $n_X := (\sum_{i=1}^m n_i)/m$  and  $u_j := n_j - n_X$ , then:

$$n_X := n_T + l/m, \quad (3.12)$$

with  $l \in \{0, \dots, m-1\}$ ,  $n_T \in \mathbb{Z}$  and  $\sum_{j=1}^m u_j = 0$ . The last condition makes possible to interpret  $\Lambda = \sum_{i=1}^m u_i \epsilon_i$  as a weight of  $su(m)$ , where  $\epsilon_1, \dots, \epsilon_m$  is an orthonormal basis of the Euclidean space  $\mathbb{R}^m$  [30, 28, 29]. In terms of Dynkin labels  $\Lambda$  can be rewritten as  $\Lambda = \sum_{j=1}^{m-1} \lambda_j \Lambda_j$ , where  $\lambda_j := u_j - u_{j+1} = n_j - n_{j+1}$ ,  $j \in \{1, \dots, m-1\}$ , are integer

numbers. Furthermore,  $(\sum_{i=1}^m i\lambda_i)/m = n_T - n_m + l/m$  and thus the  $m$ -ality of the weight  $\Lambda$  coincides with  $l/m$  in (3.12) and  $\Lambda \in \Omega_l$ . The sums in (3.11) can then be rewritten as:

$$\frac{1}{2} \sum_{i=1}^m n_i^2 = \frac{m}{2} n_X^2 + \frac{1}{2} \sum_{i=1}^m u_i^2 = \frac{m}{2} n_X^2 + h_\Lambda, \tag{3.13}$$

where:

$$h_\Lambda := \frac{1}{2} |\Lambda|^2 = \frac{1}{2} \sum_{i=1}^m u_i^2 \tag{3.14}$$

and

$$\sum_{j=1}^m \left(w - \frac{j}{m}\right) n_j = n_X \left(mw - \frac{m+1}{2}\right) - \frac{1}{m} \sum_{j=1}^m j u_j. \tag{3.15}$$

By recalling that, in terms of the fundamental weights of  $su(m)$ ,  $\rho$  has the expansion  $\rho = \sum_{j=1}^{m-1} \Lambda_j$  one gets in the  $\epsilon_1, \dots, \epsilon_m$  basis:

$$\rho = \sum_{i=1}^m \left(\frac{m+1}{2} - i\right) \epsilon_i, \tag{3.16}$$

and so:

$$\frac{1}{m} (\Lambda, \rho) = \frac{1}{m} \sum_{j=1}^m \left(\frac{m+1}{2} - j\right) u_j = -\frac{1}{m} \sum_{j=1}^m j u_j. \tag{3.17}$$

Using the above identities, (3.11) can be rewritten as:

$$\prod_{v=1}^m \Theta_3 \left(w - \frac{v}{m} | \tau\right) = \sum_{l=0}^{m-1} \left\{ \left[ \sum_{n_T \in \mathbb{Z}} q^{\frac{m}{2}(n_T + \frac{l}{m})^2} e^{2i\pi(mw - \frac{m+1}{2})(n_T + \frac{l}{m})} \right] \left[ \sum_{\Lambda \in \Omega_l} q^{h_\Lambda} e^{2i\pi(\Lambda, \frac{\rho}{m})} \right] \right\}. \tag{3.18}$$

Furthermore, by expressing this last factor in terms of  $\Theta$ -functions with characteristics:

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (w | \tau) := \sum_{u \in \mathbb{Z}} e^{\pi i \tau (u+a)^2 + 2\pi i (w+b)(u+a)} \tag{3.19}$$

and substituting the result in (3.5), it holds:

$$\frac{\eta(\tau)}{\eta(m\tau)} = \sum_{l=0}^{m-1} G_l^{(m)}(\tau) \left( \frac{1}{\eta(\tau)^{m-1}} \sum_{\Lambda \in \Omega_l} q^{h_\Lambda} e^{2i\pi(\Lambda, \frac{\rho}{m})} \right), \tag{3.20}$$

where:

$$G_l^{(m)}(\tau) := \Theta \left[ \begin{matrix} \frac{l}{m} \\ -\frac{(m+1)}{2} \end{matrix} \right] (mw | m\tau) / \Theta \left[ \begin{matrix} 0 \\ -\frac{(m+1)}{2} \end{matrix} \right] (mw | m\tau). \tag{3.21}$$

V. Kac and D. Peterson [31] have shown that for any level 1 simply laced affine Lie algebra  $X_a^{(1)}$ , all the non-zero string functions coincide with  $1/\eta(\tau)^a$  and thus the characters of the fundamental representations of  $\widehat{su(m)}_1$  read as:

$$\chi_{\widehat{\Lambda}_l}^{\widehat{su(m)}_1}(\xi | \tau) = \frac{1}{\eta(\tau)^{m-1}} \sum_{\Lambda \in \Omega_l} q^{h_\Lambda} e^{2i\pi(\Lambda, \xi)}, \tag{3.22}$$

and we finally get:

$$\frac{\eta(\tau)}{\eta(m\tau)} = \sum_{l=0}^{m-1} G_l^{(m)}(\tau) \chi_{\widehat{\Lambda}_l}^{su(m)_1} \left( \xi = \frac{\rho}{m} | \tau \right). \quad (3.23)$$

But from (3.1), we know that  $\chi_{\widehat{\Lambda}_l}^{su(m)_1}(\xi = \frac{\rho}{m} | \tau)$  is non-zero only for  $l = 0$  and thus (3.2) immediately follows, so ending the proof of Proposition 1.  $\square$

The results given in Proposition 1 are essential in order to define an extension of the chiral algebra  $\mathcal{W}_m$ .

We have shown that  $D_m$  is contained in  $\Omega_0$ , however  $\Omega_0$  has infinite weights and so it is important to look for a simpler definition of  $D_m$ . In the following, we will give a remarkable simplification characterizing  $D_m$  in terms of a finite subset of  $\Omega_0$ .

Let  $P_{m,+}$  be the subset of  $P_+$  whose weights have Dynkin labels in  $\{0, \dots, m-1\}$ , so  $P_{m,+}$  is a finite subset of  $P_+$  and  $P_+ = P_{m,+} + mP_+$ . That is, any weight  $\Lambda \in P_+$  has the form  $\Lambda' + m\Lambda''$  where  $\Lambda' \in P_{m,+}$  and  $\Lambda'' \in P_+$ ,  $\Lambda'$  being the module  $m$  part of  $\Lambda$ .

**Proposition 2.** *The identities of Proposition 1 can be written in the following more explicit forms:*

For  $m$  odd:

$$\chi_{\widehat{\Lambda}_l}^{su(m)_1} \left( \xi = \frac{\rho}{m} | \tau \right) = \delta_{l,0} \frac{\eta(\tau)}{\eta(m\tau)} = \sum_{\Lambda'' \in P_+} \sum_{\Lambda' \in P_{m,+} \cap D_m} \epsilon(w_{\Lambda'}) \chi_{\Lambda' + m\Lambda''}^{\mathcal{W}_m}(\tau). \quad (3.24)$$

For  $m = 2n$  even:

$$\chi_{\widehat{\Lambda}_l}^{su(m)_1} \left( \xi = \frac{\rho}{m} | \tau \right) = \delta_{l,0} \frac{\eta(\tau)}{\eta(m\tau)} = \sum_{\Lambda'' \in P_+} \sum_{\Lambda' \in P_{m,+} \cap D_m} (-1)^{\sum_{i=0}^{n-1} \lambda''_{2i+1}} \epsilon(w_{\Lambda'}) \chi_{\Lambda' + m\Lambda''}^{\mathcal{W}_m}(\tau), \quad (3.25)$$

where  $\lambda''_i$  are the Dynkin labels of  $\Lambda''$ .

To prove Proposition 2 we start proving the Lemma.

**Lemma 4.** *The following equivalent characterization of  $D_m = \{\Lambda \in P_+ : \Lambda = \Lambda' + m\Lambda'' \text{ with } \Lambda' \in P_{m,+} \cap D_m \text{ and } \Lambda'' \in P_+\}$  holds. Furthermore, for every weight  $\Lambda \in P_+$  of the form  $\Lambda' + m\Lambda''$ , where  $\Lambda' \in P_{m,+}$  and  $\Lambda'' \in P_+$ , it results:*

$$\chi_{\Lambda}^{su(m)} \left( \xi = \frac{\rho}{m} \right) = \chi_{\Lambda'}^{su(m)} \left( \xi = \frac{\rho}{m} \right) \quad \text{for } m \text{ odd} \quad (3.26)$$

and

$$\chi_{\Lambda}^{su(m)} \left( \xi = \frac{\rho}{m} \right) = (-1)^{\sum_{i=0}^{n-1} \lambda''_{2i+1}} \chi_{\Lambda'}^{su(m)} \left( \xi = \frac{\rho}{m} \right) \quad \text{for } m = 2n \text{ even.} \quad (3.27)$$

*Proof.* Kostant's Proposition 2 implies that  $D_m$  is the subset of the dominant weights  $P_+$  that give a non-zero value of  $\chi_{\Lambda}^{su(m)}(\xi = \frac{\rho}{m})$ . Thus, the above characterization of  $D_m$  follows by the proof of the identities (3.26) and (3.27). Indeed, they imply that  $\chi_{\Lambda}^{su(m)}(\xi = \frac{\rho}{m}) \neq 0$

if and only if  $\chi_{\Lambda'}^{su(m)}(\xi = \frac{\rho}{m}) \neq 0$ , that is  $\Lambda(=\Lambda' + m\Lambda'') \in D_m$  if and only if  $\Lambda' \in P_{m,+} \cap D_m$  and  $\Lambda'' \in P_+$ . Whenever the character of  $su(m)$  is evaluated at  $\xi \propto \rho$ , it is possible to use the following expression:

$$\chi_{\Lambda}^{su(m)}(\xi = \frac{\rho}{m}) = \prod_{\alpha > 0} \frac{\sin \frac{\pi}{m}(\alpha, (\Lambda + \rho))}{\sin \frac{\pi}{m}(\alpha, \rho)} = \prod_{i=1}^{m-1} \prod_{h=0}^{m-1-i} \frac{\sin \frac{\pi}{m}(\lambda_i + \dots + \lambda_{i+h} + h + 1)}{\sin \frac{\pi}{m}(h + 1)}. \tag{3.28}$$

Writing  $\Lambda = \Lambda' + m\Lambda''$  and using the expansion of  $\sin(a + b)$ , it follows:

$$\chi_{\Lambda}^{su(m)}\left(\xi = \frac{\rho}{m}\right) = (-1)^{a(m, \Lambda'')} \chi_{\Lambda'}^{su(m)}\left(\xi = \frac{\rho}{m}\right), \tag{3.29}$$

where  $a(m, \Lambda'') = \sum_{i=1}^{m-1} \sum_{h=0}^{m-1-i} (\lambda_i'' + \dots + \lambda_{i+h}'')$ , in terms of the Dynkin labels  $[\lambda_1'', \dots, \lambda_{m-1}'']$  of  $\Lambda''$ . Such exponent can be rewritten as  $a(m, \Lambda'') = \sum_{i=1}^{m-1} n(m, i)\lambda_i''$ , where  $n(m, i) := i(m - i)$ .

Thus, for  $m$  odd all the  $n(m, i)$  are even integers, while for  $m$  even  $n(m, i)$  are even for  $i$  even and odd for  $i$  odd. So, (3.29) implies (3.26) and (3.27), ending the proof of Lemma 4.  $\square$

It is worth pointing out that  $P_{m,+} \cap D_m$  is a finite subset of the dominant weights with zero  $m$ -ality, which implies the announced simplification in the characterization of  $D_m$ .

*Proof. of Proposition 2* The proof is now an immediate consequence of Proposition 1 and Lemma 4. Indeed, by substituting identities (3.26) and (3.27) in (3.1), the equations (3.24) and (3.25) of Proposition 2 follow.  $\square$

Finally, we use the identity (3.2) to derive the last result of this section.

**Corollary 1.** *The following identity holds:*

$$\frac{\eta(\tau)}{\eta(\tau/m)} = F_{twist}^{(m)}(\tau) \sum_{l=0}^{m-1} \mathbf{S}_{0,l}^{\widehat{su(m)}_1} \chi_{\widehat{\Lambda}_l}^{\widehat{su(m)}_1} \left(\xi = \frac{\rho}{m} \tau | \tau\right), \tag{3.30}$$

where  $\mathbf{S}_{a,b}^{\widehat{su(m)}_1}$ ,  $a, b \in \{0, \dots, m - 1\}$ , are the elements of the unitary matrix  $\mathbf{S}^{\widehat{su(m)}_1}$  that define the action of the modular transformation  $S : \tau \rightarrow -1/\tau$  on the characters of  $\widehat{su(m)}_1$ .

The explicit expression is [29]:

$$\mathbf{S}_{a,b}^{\widehat{su(m)}_1} = i^{|\Delta_+|} \left(\det A^{su(m)}\right)^{-1/2} (1+m)^{-(m^2-1)} \sum_{w \in W} \epsilon(w) e^{-2\pi i(w(\Lambda_a + \rho), \Lambda_b + \rho)/(m+1)}, \tag{3.31}$$

where  $|\Delta_+|$  is the number of positive roots,  $A^{su(m)}$  is the Cartan matrix,  $W$  is the Weyl group,  $\Lambda_a$  and  $\Lambda_b$  are the fundamental weights of  $su(m)$  and

$$F_{twist}^{(m)}(\tau) = \frac{1}{\sqrt{m}} e^{2\pi i \left(\frac{m^2-1}{24m}\right) \tau}. \tag{3.32}$$

*Proof.* The Corollary is a direct consequence of (3.2) and of the well known action of the modular transformation  $S : \tau \rightarrow -1/\tau$  on the characters of  $\widehat{su(m)}_1$  and on  $\eta(\tau)$ . By definition of  $S$  and by using (3.2), it results:  $S(\eta(\tau)/\eta(m\tau)) := \eta(-1/\tau)/\eta(-m/\tau)$  and

$S(\eta(\tau)/\eta(m\tau)) = \chi_{\widehat{\Lambda_0}}^{\widehat{su(m)}_1}(\xi = \rho/m| - 1/\tau)$ , respectively. Expanding the right hand side of these last two equalities, we obtain our result (3.30) with the following expression for  $F_{twist}^{(m)}(\tau)$ :

$$F_{twist}^{(m)}(\tau) = \frac{1}{\sqrt{m}} e^{2\pi i(\frac{1}{2}\tau|\rho|^2)/m^2}. \quad (3.33)$$

Finally, by using the Freudental-de Vries strange formula  $|\rho|^2 = (m/12) \dim su(m) = m(m^2 - 1)/12$ ,  $F_{twist}^{(m)}(\tau)$  takes the expression (3.32).  $\square$

#### 4. The $\Gamma_\theta$ -RCFT $\widehat{u(1)}_{\mathbf{K}_{m,p}}$

In order to make more clear the derivation of our theory TM, we introduce here the  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ .

As it is well known, the  $m$ -component free boson  $\widehat{u(1)}^{\otimes m}$ , with chiral algebra  $\mathfrak{A}(\widehat{u(1)})^{\otimes m}$  (the tensor product of  $m$  Heisenberg algebras), is not a rational CFT. Let  $\mathbf{K}_{m,p}$  be the  $m \times m$  symmetric matrix with integer entries:

$$\mathbf{K}_{m,p} = \mathbf{1}_{m \times m} + 2p\mathbf{C}_{m \times m}, \quad (4.1)$$

where  $p \in \mathbb{Z}$  and  $\mathbf{C}_{m \times m}$  is the  $m \times m$  matrix, all elements of which are equal to 1. An RCFT can be defined now imposing on  $\widehat{u(1)}^{\otimes m}$  the following compactification condition fixed by  $\mathbf{K}_{m,p}$  [20, 12]:

$$\varphi(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \varphi(z, \bar{z}) + 2\pi\mathbf{R}_{m,p}\mathbf{h}, \quad (4.2)$$

where  $\mathbf{h}^T := (h_1, \dots, h_m) \in \mathbb{Z}^m$  is the winding vector,  $\varphi(z, \bar{z})$  is defined by  $\varphi(z, \bar{z})^T := (\varphi^{(1)}(z, \bar{z}), \dots, \varphi^{(m)}(z, \bar{z}))$ , with  $\varphi^{(i)}(z, \bar{z})$  free boson fields, and  $\mathbf{R}_{m,p}$  is the  $m \times m$  matrix defined by<sup>6</sup>:

$$\mathbf{R}_{m,p}^T \mathbf{R}_{m,p} = \mathbf{K}_{m,p}, \quad (4.3)$$

that explicitly reads:

$$\mathbf{R}_{m,p} = \mathbf{1}_{m \times m} + \frac{1}{m} \left( \sqrt{2pm + 1} - 1 \right) \mathbf{C}_{m \times m}. \quad (4.4)$$

The compactification condition (4.2) for diagonal  $\mathbf{h} = h\mathbf{t}$  defines the following ones:

$$\varphi^{(i)}(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \varphi^{(i)}(z, \bar{z}) + 2\pi rh, \quad (4.5)$$

for the free boson fields  $\varphi^{(i)}(z, \bar{z})$  with  $h \in \mathbb{Z}$ , where the square of the compactification radius  $r$  is an odd number,  $r^2 = 2pm + 1$ .

The compactification condition has the effect to influence the zero-modes  $\mathbf{a}_0 := (a_0^{(1)}, \dots, a_0^{(m)})$  of the free boson fields only. In particular, to obtain well defined vertex operators,

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<sup>6</sup> $\mathbf{R}_{m,p}$  is the positive root of  $\mathbf{K}_{m,p}$ , which is well defined because  $\mathbf{K}_{m,p}$  is a symmetric and positive definite  $m \times m$  matrix.

under the compactification condition, the possible eigenvalues of  $\mathbf{a}_0$  are restricted to the following values  $\alpha_{\mathbf{p}} = \mathbf{p}^T \mathbf{R}_{m,p}^{-1}$  with  $\mathbf{p} \in \mathbb{Z}^m$  and

$$\mathbf{R}_{m,p}^{-1} = \mathbf{1}_{m \times m} + \frac{1}{m} \left( \frac{1}{\sqrt{2pm+1}} - 1 \right) \mathbf{C}_{m \times m}. \quad (4.6)$$

So, the h.w. vectors of the compactified  $\widehat{u(1)}^{\otimes m}$  are  $|\alpha_{\mathbf{p}}\rangle := \bigotimes_{i=1}^m |\alpha_{\mathbf{p}}^{(i)}\rangle$ , where  $\alpha_{\mathbf{p}} = (\alpha_{\mathbf{p}}^{(1)}, \dots, \alpha_{\mathbf{p}}^{(m)})$  and  $|\alpha_{\mathbf{p}}^{(i)}\rangle$  are the h.w. vectors of the  $i^{\text{th}}$  Heisenberg algebra  $\{a_k^{(i)}\}_{k \in \mathbb{Z}}$ , corresponding to the eigenvalue  $(\alpha_{\mathbf{p}})^{(i)}$  of  $a_0^{(i)}$ , i.e.

$$a_0^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle = \alpha_{\mathbf{p}}^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle, \quad a_r^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle = 0 \quad \text{for } r > 0. \quad (4.7)$$

The irreducible module corresponding to  $|\alpha_{\mathbf{p}}\rangle$  is denoted by  $H_{\mathbf{p}} := \bigotimes_{i=1}^m H_{\mathbf{p}}^{(i)}$ , where  $H_{\mathbf{p}}^{(i)}$  is the irreducible module of  $\{a_k^{(i)}\}_{k \in \mathbb{Z}}$ , defined by:

$$H_{\mathbf{p}}^{(i)} := \{a_{-n_q}^{(i)m_q} \dots a_{-n_1}^{(i)m_1} |\alpha_{\mathbf{p}}^{(i)}\rangle \quad \text{with } n_h > 0, m_h > 0, q > 0\}. \quad (4.8)$$

Moreover, the vector  $|\alpha_{\mathbf{p}}^{(i)}\rangle$  and the module  $H_{\mathbf{p}}^{(i)}$  are a h.w. vector and the corresponding irreducible module with respect to the  $c = 1$  Virasoro algebra  $\{L_n^{(i)}\}_{n \in \mathbb{Z}}$  generated by the Heisenberg algebra  $\{a_k^{(i)}\}_{k \in \mathbb{Z}}$  (see equation (B.5)), i.e.

$$L_0^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle = h_{\mathbf{p}}^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle \quad \text{with } h_{\mathbf{p}}^{(i)} = \frac{1}{2} \alpha_{\mathbf{p}}^{(i)2}, \quad L_n^{(i)} |\alpha_{\mathbf{p}}^{(i)}\rangle = 0 \quad \text{for } n > 0. \quad (4.9)$$

The zero mode of  $m$  independent  $c = 1$  Virasoro algebras  $\{L_n^{(i)}\}_{n \in \mathbb{Z}}$  is defined by:

$$L_0 := \sum_{i=1}^m L_0^{(i)}, \quad (4.10)$$

and so we have:

$$L_0 |\alpha_{\mathbf{p}}\rangle = h_{\mathbf{p}} |\alpha_{\mathbf{p}}\rangle \quad (4.11)$$

with

$$h_{\mathbf{p}} = \sum_{i=1}^m h_{\mathbf{p}}^{(i)} = \alpha_{\mathbf{p}} \alpha_{\mathbf{p}}^T / 2. \quad (4.12)$$

The corresponding character is:

$$\text{Tr}_{H_{\mathbf{p}}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \frac{1}{\eta(\tau)^m} q^{h_{\mathbf{p}}} e^{2\pi i w \sqrt{2mp+1} \alpha_{\mathbf{p}} \mathbf{t}}, \quad (4.13)$$

where  $J := \mathbf{a}_0 \mathbf{t} \det(\mathbf{R}_{m,p})$  is the conformal charge and by definition  $H_{\mathbf{p}}$  is the eigenspace of  $J$  corresponding to the eigenvalue  $\sqrt{2mp+1} \alpha_{\mathbf{p}} \mathbf{t}$ .

As in the case of the single free boson CFT (see appendix B), we define the chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  extension of  $\mathfrak{A}(\widehat{u(1)})^{\otimes m}$  by adding to it the modes of the two chiral currents:

$$\Gamma_{\mathbf{K}_{m,p}}^{\pm}(z) := : e^{\pm i \mathbf{t}^T \mathbf{R}_{m,p} \phi(z)} :, \quad (4.14)$$

where  $\phi(z)^T := (\phi^{(1)}(z), \dots, \phi^{(m)}(z))$  is the chiral part of  $\varphi(z, \bar{z})^T$ . By definition:

$$\Gamma_{\mathbf{K}_{m,p}}^{\pm}(z) = \bigotimes_{i=1}^m \Gamma_{(2mp+1)}^{(i)\pm}(z) \quad (4.15)$$

where:

$$\Gamma_{(2mp+1)}^{(i)\pm}(z) := e^{\pm i\sqrt{2mp+1}\phi^{(i)}(z)} \quad (4.16)$$

are locally anticommuting Fermi fields, with half integer  $(2mp+1)/2$  conformal dimensions.

Let  $\{k_i\}_{i \in \{1, \dots, a\}} \in \mathbb{Z}_+$  and  $\{\mathbf{u}_{i,j}\}_{j \in \{1, \dots, d_i\}, i \in \{1, \dots, a\}} \in \mathbb{Z}^m$  be the eigenvalues and a basis in  $\mathbb{Z}^m$  of the corresponding eigenvectors of  $\mathbf{K}$ ,  $\mathbf{K}\bar{\mathbf{u}}_{i,j} = k_i\mathbf{u}_{i,j}$ , respectively. Then:

$$\mathbf{K}\mathbb{Z}^m := \left\{ \mathbf{p} \in \mathbb{Z}^m : \mathbf{p} = \sum_{i=1}^a \sum_{j=1}^{d_i} c_{i,j} \mathbf{u}_{i,j} \text{ with } c_{i,j} \in \mathbb{Z} \right\} \quad (4.17)$$

and the quotient  $\mathbb{Z}_{\mathbf{K}} := \mathbb{Z}^m / \mathbf{K}\mathbb{Z}^m$  is:

$$\mathbb{Z}_{\mathbf{K}} := \left\{ \mathbf{p} \in \mathbb{Z}^m : \mathbf{p} = \sum_{i=1}^a \sum_{j=1}^{d_i} c_{i,j} \mathbf{u}_{i,j} \text{ with } c_{i,j} \in \{0, \dots, k_i - 1\} \right\}. \quad (4.18)$$

The matrix  $\mathbf{K}_{m,p}$  has two distinct eigenvalues:  $k_1 = 2mp + 1$ , with degeneracy 1 and eigenvector  $\mathbf{t}$ , and  $k_2 = 1$ , with degeneracy  $m - 1$  and  $m - 1$  independent eigenvectors  $\{\mathbf{u}_j\}_{j \in \{1, \dots, m-1\}} \in \mathbb{Z}^m$ , simply characterized by the orthogonality condition

$$\mathbf{u}_j^T \mathbf{t} = 0, \quad (4.19)$$

so that:

$$\mathbb{Z}_{\mathbf{K}_{m,p}} := \{\mathbf{p} \in \mathbb{Z}^m : \mathbf{p} = b\mathbf{t} \text{ with } b \in \{0, \dots, 2mp\}\}. \quad (4.20)$$

Now,  $\det(\mathbf{R}_{m,p}^2) = 2mp + 1$  and the chiral algebra  $\mathfrak{A}(\widehat{\mathfrak{u}(1)}_{\mathbf{K}_{m,p}})$  has h.w. vectors  $|\alpha_{bt}\rangle$  with conformal weights

$$\tilde{h}_b := \frac{mb^2}{2(2mp+1)}, \quad (4.21)$$

corresponding to  $\alpha_{bt} = b\mathbf{t}^T \mathbf{R}_{m,p}^{-1}$  with  $b\mathbf{t} \in \mathbb{Z}_{\mathbf{K}_{m,p}}$ . The related irreducible modules are:

$$H_b := \bigoplus_{\mathbf{q} \in \mathbb{Z}^m} H_{b\mathbf{t} + \mathbf{K}_{m,p}\mathbf{q}}, \quad (4.22)$$

with  $b \in \{0, \dots, 2mp\}$ , and so the characters are:

$$\tilde{\chi}_b(w|\tau) := \text{Tr}_{H_b} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right). \quad (4.23)$$

More explicitly, they are given by:

$$\begin{aligned} \tilde{\chi}_b(w|\tau) &= \frac{1}{\eta(\tau)^m} \times \\ &\times \sum_{\mathbf{q} \in \mathbb{Z}^m} e^{2\pi i \left\{ \frac{\tau}{2} \left[ (b\mathbf{t}^T \mathbf{R}_{m,p}^{-1} + \mathbf{q}^T \mathbf{R}_{m,p}) (b\mathbf{t}^T \mathbf{R}_{m,p}^{-1} + \mathbf{q}^T \mathbf{R}_{m,p})^T \right] + w \det(\mathbf{R}_{m,p}) (b\mathbf{t}^T \mathbf{R}_{m,p}^{-1} + \mathbf{q}^T \mathbf{R}_{m,p}) \mathbf{t} \right\}}. \end{aligned} \quad (4.24)$$



The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  defines a  $\Gamma_\theta$ -RCFT because its characters  $\tilde{\chi}_b(w|\tau)$  define a  $(2mp+1)$ -dimensional representation of the modular subgroup  $\Gamma_\theta$ :

The transformation  $T^2$ :

$$\tilde{\chi}_b(w|\tau+2) = e^{i2\pi[2(\tilde{h}_b - \frac{m}{24})]} \tilde{\chi}_b(w|\tau), \quad (4.25)$$

where  $(\tilde{h}_b - m/24)$  is the modular anomaly of a h.w. representation of conformal dimension  $\tilde{h}_b$  in a  $\Gamma_\theta$ -RCFT with central charge  $c = m$ .

The transformation  $S$ :

$$\tilde{\chi}_b\left(\frac{w}{\tau} \middle| -\frac{1}{\tau}\right) = \frac{1}{\sqrt{2pm+1}} \sum_{b'=0}^{2pm} e^{\frac{2i\pi m b b'}{2pm+1}} \tilde{\chi}_{b'}(w|\tau). \quad (4.26)$$

Such modular transformations can be simply derived from those of the  $\Theta$ -functions with characteristics. We denote this  $\Gamma_\theta$ -RCFT simply with  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ .

The fact that the matrix  $\mathbf{K}_{m,p}$  is symmetric implies that  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  is invariant under the exchange of a pair of free bosons. More precisely, the exchange of a pair of free bosons is an outer automorphism on the chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$ . Let  $g$  be defined as the element that acts on  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  bringing the field in position  $i$  into that in position  $i+1$ ,  $i \in \{1, \dots, m\}$ , with the periodicity condition  $m+1 \equiv 1$ . Then,  $g$  is an outer automorphism of  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$ . We observe that  $g^m \equiv \mathbf{1}$  and  $g^h \neq \mathbf{1}$  for  $h \in \{1, \dots, m-1\}$ , so  $g$  generates the discrete symmetry group  $\mathbb{Z}_m$  of outer automorphisms of  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$ . The cyclic permutation orbifold in the next section is made with respect to this discrete symmetry group  $\mathbb{Z}_m$  of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ .

**Proposition 3.** *The h.w. representations of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  can be expressed in terms of those of the tensor product  $\widehat{u(1)}_{m(2mp+1)} \otimes \widehat{su(m)}_1$ , as the following character decompositions show:*

$$\tilde{\chi}_b(w|\tau) = \sum_{l=0}^{m-1} \chi_l^{\widehat{su(m)}_1}(\tau) K_{(2mp+1)l+mb}^{(m(2mp+1))}(w|\tau), \quad (4.27)$$

with  $b \in \{0, \dots, 2mp\}$  and  $K_b^{(q)}(w|\tau)$  the characters of the  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_q$ ,  $q$  odd, given by:

$$K_b^{(q)}(w|\tau) = \frac{1}{\eta(\tau)} \Theta \left[ \begin{matrix} \frac{b}{q} \\ 0 \end{matrix} \right] (qw|q\tau). \quad (4.28)$$

*Proof.* It results:

$$\alpha_{b,\mathbf{q}} := \alpha_{\mathbf{b}\mathbf{t} + \mathbf{K}_{m,p}\mathbf{q}} = \sqrt{2pm+1} \left[ \frac{b}{2pm+1} + \left( \frac{l}{m} + q \right) \right] \mathbf{t}^T + \mathbf{u}^T, \quad (4.29)$$

where  $l/m + q := \left( \sum_{i=1}^{m-1} q_i \right) / m$ ,  $l \in \{0, \dots, m-1\}$ ,  $\mathbf{u} := \mathbf{q} - \mathbf{t} \left( \sum_{i=1}^{m-1} q_i \right) / m$  and  $\mathbf{q}^T := (q_1, \dots, q_m) \in \mathbb{Z}^m$ . We observe that, denoting  $\alpha_{b,\mathbf{q}} := (\alpha_{b,\mathbf{q}}^{(1)}, \dots, \alpha_{b,\mathbf{q}}^{(m)})$ , it follows:

$$\alpha_{b,\mathbf{q}}^{(i)} - \alpha_{b,\mathbf{q}}^{(j)} = q_i - q_j \in \mathbb{Z} \quad \forall i, j \in \{1, \dots, m\}. \quad (4.30)$$

Such a result is at the origin of the decomposition of the modules  $H_b$  of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  in terms of the tensor product of those of the level 1 affine Lie algebra  $\widehat{su(m)}_1$  and of the free boson  $\widehat{u(1)}_{m(2pm+1)}$ .

Indeed, to each  $\alpha_{b,\mathbf{q}}$  corresponds the weight  $\Lambda := \sum_{i=1}^{m-1} \lambda_i \Lambda_i$  of  $su(m)$ , where  $\Lambda_i$  are the fundamental weights of  $su(m)$  and  $\lambda_i$  are the Dynkin labels,  $\lambda_i := q_i - q_{i+1} \in \mathbb{Z}$ . The  $m$ -ality of the weight  $\Lambda$  coincides by definition with  $l/m$ . We observe that when  $\mathbf{q}$  spans  $\mathbb{Z}^m$  then  $q$  spans  $\mathbb{Z}$ ,  $l$  spans  $\{0, \dots, m-1\}$  and for any fixed  $l$ ,  $\Lambda$  spans  $\Omega_l$ .

By using the orthogonality condition (4.19) and the definition (4.29), we obtain:

$$\alpha_{b,\mathbf{q}\mathbf{t}} = m\sqrt{2mp+1} \left( \frac{mb + (2pm+1)l}{m(2pm+1)} + q \right); \tag{4.31}$$

so, the conformal dimension  $h_{b,\mathbf{q}} := \alpha_{b,\mathbf{q}} \alpha_{b,\mathbf{q}}^T / 2$  has the following expression in terms of  $b$ ,  $l$ ,  $q$  and  $\Lambda$ :

$$h_{b,\mathbf{q}} = \frac{m(2pm+1)}{2} \left( \frac{mb + (2pm+1)l}{m(2pm+1)} + q \right)^2 + h_\Lambda, \tag{4.32}$$

where  $h_\Lambda$  is the conformal dimension (3.14) with  $u_i := q_i - \left( \sum_{j=1}^m q_j \right) / m$ . The characters of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  can then be written in the form:

$$\begin{aligned} \tilde{\chi}_b(w|\tau) &= \sum_{l=0}^{m-1} \left( \frac{1}{\eta(\tau)^{m-1}} \sum_{\Lambda \in \Omega_l} q^{h_\Lambda} \right) \cdot \\ &\cdot \left( \sum_{q \in \mathbb{Z}} \frac{e^{2\pi i \left\{ \tau \frac{m(2pm+1)}{2} \left( \frac{mb + (2pm+1)l}{m(2pm+1)} + q \right)^2 + wm(2mp+1) \left( \frac{mb + (2pm+1)l}{m(2pm+1)} + q \right) \right\}}}{\eta(\tau)} \right) \end{aligned} \tag{4.33}$$

which, by (3.22) and by the definition of the characters  $K_a^{(m(2pm+1))}(w|\tau)$  of  $\widehat{u(1)}_{m(2pm+1)}$ , coincides with (4.27).  $\square$

**Proposition 4.**  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  can be seen as a  $\Gamma_\theta$ -RCFT extension of  $su(m) \otimes W_{1+\infty}^{(m)}$ , as it follows by the decomposition of its characters:

$$\tilde{\chi}_b(w|\tau) = \sum_{\mathbf{q} \in \mathbb{Z}^{(m,+)}} d_{su(m)}(\Lambda) \chi_{\mathbf{r}(b,\mathbf{q})}^{\mathbf{W}_m}(w|\tau), \tag{4.34}$$

where  $\mathbf{r}(b,\mathbf{q}) := \alpha_{b,\mathbf{q}}$ ,  $\mathbb{Z}^{(m,+)} := \{ \mathbf{q} \in \mathbb{Z}^m : q_1 \geq \dots \geq q_m \}$  and  $\Lambda$  is defined for  $\mathbf{q} \in \mathbb{Z}^m$  as:

$$\Lambda := \sum_{i=1}^{m-1} \lambda_i \Lambda_i \quad \text{with:} \quad \begin{cases} \lambda_i := q_i - q_{i+1} \in \mathbb{Z} \quad \forall i \in \{1, \dots, m-1\} \\ l/m + q := \left( \sum_{i=1}^{m-1} q_i \right) / m \end{cases}, \tag{4.35}$$

and  $\Lambda_i$  the fundamental weights of  $su(m)$ .

*Proof.* The fully degenerate h.w.  $\mathbf{r}$ , that define the h.w. module  $\mathbf{W}_\mathbf{r}^{(m)}$  of  $W_{1+\infty}^{(m)}$ , corresponds to the following value of the h.w.  $\mathbf{r}(b,\mathbf{q}) := \alpha_{b,\mathbf{q}}$  with the only restriction  $\mathbf{q} \in \mathbb{Z}^{(m,+)}$ . Indeed,

by (4.29)  $\mathbf{r}(b, \mathbf{q})$  belongs to  $\mathbb{P}^{(m)}$  if and only if  $\mathbf{q} \in \mathbb{Z}^{(m,+)}$  and by the definition (2.11) and the relations (4.32) and (4.31) the corresponding character is:

$$\chi_{\mathbf{r}(b, \mathbf{q})}^{\mathbf{W}_m}(w|\tau) = \chi_{\Lambda}^{\mathcal{W}_m}(\tau) \frac{e^{2\pi i \left\{ \tau \frac{m(2pm+1)}{2} \left( \frac{mb+(2pm+1)l}{m(2pm+1)} + q \right)^2 + wm(2mp+1) \left( \frac{mb+(2pm+1)l}{m(2pm+1)} + q \right) \right\}}}{\eta(\tau)}, \quad (4.36)$$

where  $l$ ,  $q$  and  $\Lambda$  are defined by (4.35).

The identity (4.27) can be rewritten by using (2.10) as:

$$\tilde{\chi}_b(w|\tau) = \sum_{l=0}^{m-1} \sum_{\Lambda \in P_+ \cap \Omega_l} d_{su(m)}(\Lambda) \left( \chi_{\Lambda}^{\mathcal{W}_m}(\tau) K_{(2mp+1)l+mb}^{(m(2pm+1))}(w|\tau) \right), \quad (4.37)$$

and so, by (4.36) and by the definition of the characters  $K_a^{(m(2pm+1))}(w|\tau)$  of  $\widehat{u(1)}_{m(2pm+1)}$ , (4.37) coincides with (4.34).  $\square$

The above identity makes clear the meaning of the claim that  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  defines a  $\Gamma_\theta$ -RCFT extension of the chiral algebra given by the tensor product of the fully degenerate representations of  $W_{1+\infty}^{(m)}$  times the representations of  $su(m)$  specialized to  $\xi = (z\rho)|_{z=0}$ .

In particular, the module  $H_b$  of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ , corresponding to the h.w.  $\alpha_b := b\mathbf{t}$ , has the following expansion:

$$H_b = \bigoplus_{\mathbf{q} \in \mathbb{Z}^{(m,+)}} F_{su(m)}(\Lambda) \otimes \mathbb{W}_{\mathbf{r}(b, \mathbf{q})}^{(m)}, \quad (4.38)$$

where  $F_{su(m)}(\Lambda)$  is the h.w. module of  $su(m)$  corresponding to the h.w.  $\Lambda$ .

## 5. The $\mathbb{Z}_m$ -orbifold of $\widehat{u(1)}_{\mathbf{K}_{m,p}}$

In this section, we just give the essential elements to identify our TM. That is, we construct explicitly the  $\mathbb{Z}_m$  cyclic permutation orbifold of the  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ . In particular, a finite set of irreducible characters (modules) of the orbifold chiral algebra  $\mathfrak{A}_{\text{TM}} := \mathfrak{A}^{\mathbb{Z}_m}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  is found. Their modular transformations have been performed, proving that they give a unitary finite dimensional representation of the modular subgroup  $\Gamma_\theta$ , i.e. TM is a  $\Gamma_\theta$ -RCFT.

We refer to our previous paper [1] for the construction of the vertex operators (the chiral primary fields of TM) by the  $m$ -reduction procedure [32]. Furthermore, here we consider the case  $m > 2$  and prime, the particular  $m = 2$  case being developed in [1].

The orbifold construction makes possible to define new RCFTs starting from a given RCFT by quotienting it with a generic discrete symmetry group  $G$ . The discrete group  $G$  can be characterized more precisely as a group of automorphisms of the chiral algebra  $\mathfrak{A}$  of the original RCFT. The orbifold chiral algebra  $\mathfrak{A}^G := \mathfrak{A}/G$  is then defined as the subalgebra of  $\mathfrak{A}$  invariant under  $G$ . The study of the orbifolds was first introduced in the context of string theory in order to approximate CFT on Calabi-Yau manifolds [33] and further developed in [34–36]. A first detailed study of the general properties of the orbifolds was done in [11], while in [8] a complete study of orbifolds with respect to discrete groups of inner automorphisms was given.

Here, we are interested in the class of the cyclic permutation orbifolds. It was first introduced in [13, 14] on RCFTs characterized as the tensor product of  $m$  copies of a given RCFT. The subject was further developed in [15, 16, 37–41], and it was also studied by using the approach of the  $m$ -reduction technique in [32, 42, 1]. A more complete classification has been recently presented in [43], where the orbifold construction is applied in the more general framework of the lattice vertex algebras (which are not simply tensor products of RCFTs). In particular, the twisted vertex operator algebras and their modules are studied.

Let us recall the main steps of the procedure to build the finite set of irreducible representations of a  $\mathbb{Z}_m$  cyclic permutation orbifold. The orbifold chiral algebra  $\mathfrak{A}^{\mathbb{Z}_m}$  has a finite set of irreducible representations that splits in two sectors, untwisted and twisted one. The irreducible representations of the untwisted sector are generated by restricting those of the original chiral algebra  $\mathfrak{A}$  to their invariant part with respect to the elements of  $\mathbb{Z}_m$ . The characters of the untwisted irreducible  $\mathfrak{A}^{\mathbb{Z}_m}$ -representations are not anymore closed under modular transformations. Then, the irreducible  $\mathfrak{A}^{\mathbb{Z}_m}$ -representations of the twisted sector are generated by applying to the untwisted irreducible  $\mathfrak{A}^{\mathbb{Z}_m}$ -characters the modular transformations<sup>7</sup>,  $T^j S \in PSL(2, \mathbb{Z})$  with  $j \in \{0, \dots, m-1\}$ .

It is worth pointing out that our theory TM defines a family of lattice orbifolds which can be included in the general classification presented<sup>8</sup> in [43]. Indeed, TM describes the cyclic permutation orbifolds with respect to the outer automorphisms  $\mathbb{Z}_m$  of the lattice vertex algebras  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$ , where any  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  is not a simple tensor product of  $m$  copies of a  $\Gamma_\theta$ -RCFT.

**Proposition 5.** *The theory TM, characterized as the  $\mathbb{Z}_m$ -orbifold of the  $\Gamma_\theta$ -RCFT  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ , has the following content:*

**The untwisted sector.** *The so called “P-P” untwisted sector of TM coincides with the  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ ; so, it has  $2pm + 1$  h.w. representations with conformal dimensions:*

$$\tilde{h}_{(b,(1,1))} = \frac{mb^2}{2(2pm + 1)} = \tilde{h}_b, \tag{5.1}$$

where  $b = 0, \dots, 2mp$  and corresponding characters:

$$\tilde{\chi}_{(b,(1,1))}(w|\tau) = \tilde{\chi}_b(w|\tau). \tag{5.2}$$

*The so called “P-A” untwisted sector of TM has  $2pm + 1$  h.w. representations, each one with degeneracy  $m - 1$  and conformal dimension:*

$$\tilde{h}_{(b,(1,g^i))} = \frac{mb^2}{2(2pm + 1)}, \tag{5.3}$$

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<sup>7</sup>In our case, the theory to which is applied the orbifold is a  $\Gamma_\theta$ -RCFT and  $m$  is a prime number, so we apply  $T^{2j} S \in \Gamma_\theta$  with  $j \in \{0, \dots, m-1\}$ .

<sup>8</sup>In particular, the  $m$ -reduction generates the vertex operators of our twisted sector, which can be included into the class of twist fields defined in [43].

where  $b = 0, \dots, 2mp$ ,  $i = 1, \dots, m - 1$  and  $g$  is the generator of the discrete group  $\mathbb{Z}_m$ . The corresponding characters are:

$$\tilde{\chi}_{(b,(\mathbf{1},g^i))}(w|\tau) = K_b^{(2pm+1)}(mw|m\tau). \tag{5.4}$$

**The twisted sector.** The so called “A-P” twisted sector of TM has  $m(2pm + 1)$  h.w. representations each one with degeneracy  $m - 1$  and conformal dimension:

$$h_{(s,f,i)} = \frac{s^2}{2(2pm + 1)m} + \frac{m^2 - 1}{24m} + \frac{f}{2m}, \tag{5.5}$$

where  $s = 0, \dots, 2mp$ ,  $f = 0, \dots, m - 1$  and  $i = 1, \dots, m - 1$ . Here, the term  $(m^2 - 1) / 24m$  takes into account the conformal dimension of the twist. The characters are:

$$\chi_{(s,f,i)}(w|\tau) = \frac{1}{m} \sum_{j=0}^{m-1} e^{-2\pi i(2j)(h_{(s,f,i)} - \frac{m}{24})} K_s^{(2pm+1)} \left( w \left| \frac{\tau + 2j}{m} \right. \right). \tag{5.6}$$

*Proof.* Here, we construct all the sectors of the  $\mathbb{Z}_m$ -orbifold of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  showing that they coincide with those written above in Proposition 5.

**The untwisted sector** of the orbifold is obtained introducing a new h.w. vector  $|\alpha_{bt}, (\mathbf{1}, \pi)\rangle$  and module  $H_b^{(\mathbf{1}, \pi)}$  for any element  $\pi \in \mathbb{Z}_m$  and any h.w. vector<sup>9</sup>  $|\alpha_{bt}\rangle$  and module  $H_b$  in the native theory  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ . The new module  $H_b^{(\mathbf{1}, \pi)}$  is defined selecting out from the module  $H_b$  only the vectors that are invariant under the action of  $\pi$ .

We observe that by the definition of  $g$  a vector  $|\alpha\rangle := \bigotimes_{i=1}^m |\alpha^{(i)}\rangle$  is invariant under the action of  $\pi = g^i$ ,  $i \in \{1, \dots, m - 1\}$ , if and only if it is a diagonal vector,  $\alpha = a \mathbf{t}^T$ .

Thus, the irreducible modules  $H_{bt+\mathbf{K}_{m,p}\mathbf{q}}$  of  $\mathfrak{A}(\widehat{u(1)})^{\otimes m}$ , which can participate to build  $H_b^{(\mathbf{1}, g^i)}$   $i \in \{1, \dots, m - 1\}$ , are only those with diagonal h.w. vector  $|\alpha_{bt+\mathbf{K}_{m,p}\mathbf{q}}\rangle$ . By (4.29),  $|\alpha_{bt+\mathbf{K}_{m,p}\mathbf{q}}\rangle$  is diagonal if and only if  $\mathbf{q}$  is diagonal,  $\mathbf{q} := q \mathbf{t}^T$ . So, the vectors in the module  $H_b^{(\mathbf{1}, g^i)}$ ,  $i \in \{1, \dots, m - 1\}$ , are only the diagonal vectors in  $H_{bt+\mathbf{K}_{m,p}q\mathbf{t}^T}$  with  $q \in \mathbb{Z}$ .

Summarizing, the modules  $H_b^{(\mathbf{1}, \pi)}$  of the untwisted sector of the  $\mathbb{Z}_m$ -orbifold of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  are defined in the following way:

$$H_b^{(\mathbf{1}, \pi)} := \begin{cases} \text{For } \pi = g^0 = \mathbf{1}, \text{ the identity: } \bigoplus_{\mathbf{q} \in \mathbb{Z}^m} H_{bt+\mathbf{K}_{m,p}\mathbf{q}} = H_b \\ \text{For } \pi = g^i \forall i \in \{1, \dots, m - 1\} : \bigoplus_{q \in \mathbb{Z}} H_{b,q}^{(g)} \end{cases}, \tag{5.7}$$

where  $H_{b,q}^{(g)}$  is the submodule of  $H_{bt+\mathbf{K}_{m,p}q\mathbf{t}^T}$  with only diagonal vectors.

The conformal weight of  $|\alpha_{bt}, (\mathbf{1}, \pi)\rangle$  is:

$$\tilde{h}_{(b,(\mathbf{1},\pi))} = \frac{mb^2}{2(2pm + 1)} \tag{5.8}$$

and the corresponding character is:

$$\tilde{\chi}_{(b,(\mathbf{1},\pi))}(w|\tau) := Tr_{H_b^{(\mathbf{1},\pi)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right). \tag{5.9}$$

---

<sup>9</sup>We observe that the h.w. vectors  $|\alpha_{bt}\rangle$  of  $\mathfrak{A}(\widehat{u(1)}_{\mathbf{K}_{m,p}})$  are invariant under the action of  $g$ .

For  $\pi = g^0 = \mathbf{1}$  (the Identity), it reads as:

$$\tilde{\chi}_{(b,(\mathbf{1},\mathbf{1}))}(w|\tau) := \text{Tr}_{H_b} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \tilde{\chi}_b(w|\tau), \quad (5.10)$$

while for  $\pi = g^i$ ,  $i \in \{1, \dots, m-1\}$ , it reads as:

$$\tilde{\chi}_{(b,(\mathbf{1},\pi))}(w|\tau) := \text{Tr}_{H_b^{(1,\pi)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) := \sum_{q \in \mathbb{Z}} \text{Tr}_{H_{b,q}^{(g)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right). \quad (5.11)$$

The definition of  $H_{b,q}^{(g)}$  implies:

$$\text{Tr}_{H_{b,q}^{(g)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \quad (5.12)$$

$$= \frac{1}{\eta(m\tau)} e^{2\pi i \left\{ \frac{\tau}{2} [(\alpha_b + q\mathbf{t}^T \mathbf{R}_{m,p})(\alpha_b + q\mathbf{t}^T \mathbf{R}_{m,p})^T] + w \det(\mathbf{R}_{m,p})(\alpha_b + q\mathbf{t}^T \mathbf{R}_{m,p})\mathbf{t} \right\}} \quad (5.13)$$

or more explicitly using (4.29), (4.32) and (4.31):

$$\text{Tr}_{H_{b,q}^{(g)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \frac{1}{\eta(m\tau)} e^{2\pi i \left\{ \tau \frac{m(2pm+1)}{2} \left( \frac{b}{2pm+1} + q \right)^2 + wm(2mp+1) \left( \frac{b}{2pm+1} + q \right) \right\}}, \quad (5.14)$$

and finally:

$$\tilde{\chi}_{(b,(\mathbf{1},\pi))}(w|\tau) = K_b^{(2pm+1)}(mw|m\tau). \quad (5.15)$$

The identity (5.10) implies that the h.w.  $(\alpha_b, (\mathbf{1}, \mathbf{1}))$  representation of the  $\mathbb{Z}_m$ -orbifold of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  coincides with the h.w.  $\alpha_b$  representation of  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ , that is the ‘‘P-P’’ sector coincides with  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$ , while the identity (5.15) implies that there is a  $m-1$  degeneracy in the other h.w. representations that define the ‘‘P-A’’ sector.

**The twisted sector** of the orbifold is generated by the action of the group  $\Gamma_\theta$  on the untwisted sector. In particular, being  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  a  $\Gamma_\theta$ -RCFT the twisted sector is generated by the action of the group  $\Gamma_\theta$  on the ‘‘P-A’’ untwisted sector.

More precisely, by means of the modular transformation  $S$  from the characters  $\tilde{\chi}_{(b,(\mathbf{1},g^i))}(w|\tau)$  of the ‘‘P-A’’ untwisted sector we can generate the characters  $\tilde{\chi}_{(b,(g^i,\mathbf{1}))}(w|\tau) = K_b^{(2pm+1)}(w|\frac{\tau}{m})$ ,  $i \in \{1, \dots, m-1\}$ , of the twisted sector.

Then, using the modular transformation  $T^2$  on the characters  $\tilde{\chi}_{(b,(g^i,\mathbf{1}))}(w|\tau)$ , the following basis in the twisted sector is obtained:

$$\tilde{\chi}_{(b,(g^i,g^{2j}))}(w|\tau) = K_b^{(2pm+1)} \left( w \left| \frac{\tau + 2j}{m} \right. \right), \quad (5.16)$$

where  $b \in \{0, \dots, 2pm\}$ ,  $i \in \{1, \dots, m-1\}$  and  $j \in \{0, \dots, m-1\}$ . Also these characters are degenerate with respect to the index  $i \in \{1, \dots, m-1\}$ .

The invertibility of the equality in (5.6) implies that the characters in (5.6) and (5.16) simply define two different basis of the same twisted sector.

The reason why we have chosen the characters  $\chi_{(s,f,i)}(w|\tau)$  as a basis is due to the fact that they correspond to well defined h.w. representations, as it can be seen by looking at the transformations of these characters under the elements of the modular subgroup  $\Gamma_\theta$ .  $\square$

**Proposition 6.** *The theory TM is a  $\Gamma_\theta$ -RCFT.*

*Proof.* We have to show that the characters of TM give a finite dimensional representation of the modular group  $\Gamma_\theta$ .

The  $m$ -component free bosons  $\widehat{u(1)}_{\mathbf{K}_{m,p}}$  is a  $\Gamma_\theta$ -RCFT. The modular transformations of the corresponding characters are given in section 4.

The modular transformations of the ‘‘P-A’’ untwisted and the ‘‘A-P’’ twisted characters are derived using their definitions (5.4), (5.6) and the known modular transformations for the free boson  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{2mp+1}$ , given in appendix B.

**The modular transformations of the characters of the ‘‘P-A’’ untwisted sector.**  
The transformation  $T^2$ :

$$\tilde{\chi}_{(b,(1,g^i))}(w|\tau + 2) = e^{i2\pi\left[2\left(\tilde{h}_{(b,(1,g^i))} - \frac{m}{24}\right)\right]} \tilde{\chi}_{(b,(1,g^i))}(w|\tau), \quad (5.17)$$

where  $\left(\tilde{h}_{(b,(1,g^i))} - m/24\right)$  is the modular anomaly of a h.w. representation of conformal dimension  $\tilde{h}_{(b,(1,g^i))}$  in a  $\Gamma_\theta$ -RCFT with central charge  $c = m$ .

The transformation  $S$ :

$$\tilde{\chi}_{(b,(1,g^i))}\left(\frac{w}{\tau} \middle| -\frac{1}{\tau}\right) = \sum_{\mu=0}^{2pm} \sum_{f=0}^{m-1} \frac{e^{\frac{2i\pi b\mu}{2mp+1}}}{\sqrt{2mp+1}} \chi_{(\mu,f,i)}(w|\tau), \quad (5.18)$$

it brings the characters of the ‘‘P-A’’ untwisted sector  $\tilde{\chi}_{(b,(1,g^i))}$  into those of the ‘‘A-P’’ twisted sector  $\chi_{(\mu,f,i)}$ .

**The modular transformations of the characters of the ‘‘A-P’’ twisted sector.**  
The transformation  $T^2$ :

$$\chi_{(s,f,i)}(w|\tau + 2) = e^{i2\pi\left[2\left(\tilde{h}_{(s,f,i)} - \frac{m}{24}\right)\right]} \chi_{(s,f,i)}(w|\tau), \quad (5.19)$$

where  $\left(\tilde{h}_{(s,f,i)} - \frac{m}{24}\right)$  is the modular anomaly of a h.w. representation of conformal dimension  $\tilde{h}_{(s,f,i)}$  in a  $\Gamma_\theta$ -RCFT with central charge  $c = m$ .

The transformation  $S$  on the characters  $\chi_{(s,f,i)}$  is the most subtle to find. By the definition (5.6) of the characters  $\chi_{(s,f,i)}(w|\tau)$  it is clear that to find their transformation under  $S$  we have to find the action of  $S$  on the characters  $K_s^{(2pm+1)}(w|(\tau + 2j)/m)$  of  $\widehat{u(1)}_{2mp+1}$ .

**Lemma 5.** *The modular transformation  $S(w|\tau) := (w/\tau | -1/\tau)$  acts on the characters  $K_s^{(2pm+1)}(w|(\tau + 2j)/m)$  of the free boson  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{2mp+1}$ , in the following way:*

$$S(K_s^{(2pm+1)}(w|\frac{\tau}{m})) = \sum_{b=0}^{2pm} \left( \frac{e^{\frac{2i\pi sb}{2mp+1}}}{\sqrt{2mp+1}} \right) K_b^{(2pm+1)}(mw|m\tau) \quad (5.20)$$

and

$$S(K_s^{(2pm+1)} \left( w \left| \frac{\tau + 2j}{m} \right. \right)) = \sum_{b=0}^{2pm} (\mathcal{A}_{(m,p,2j)})_{(s,b)} K_b^{(2pm+1)} \left( w \left| \frac{\tau + 2j^*}{m} \right. \right) \quad \forall j \in (1, \dots, m-1), \quad (5.21)$$

where  $(\mathcal{A}_{(m,p,2j)})_{(s,\mu)}$  are the entries of the  $(2mp+1) \times (2mp+1)$  matrix  $\mathcal{A}_{(m,p,2j)}$  that represents the action of the modular transformation  $A_{(m,2j)} \in \Gamma_\theta$  on the characters of  $\widehat{u(1)}_{2mp+1}$ . The matrix  $A_{(m,2j)} \in \Gamma_\theta$  and the integer number  $j^* \in (1, \dots, m-1)$  are defined in a univocal way by the conditions:

$$S \left( w \left| \frac{\tau + 2j}{m} \right. \right) := \left( \frac{w}{\tau} \left| \frac{(-1/\tau) + 2j}{m} \right. \right) = A_{(m,2j)} \left( w \left| \frac{\tau + 2j^*}{m} \right. \right), \quad (5.22)$$

for any fixed  $j \in (1, \dots, m-1)$ .

*Proof.* of Lemma 5 Relation (5.20) can be derived in the following way:

$$S((w \left| \frac{\tau}{m} \right.)) := \left( \frac{w}{\tau} \left| -\frac{1}{m\tau} \right. \right) \quad (5.23)$$

but now the right hand side can be seen as the transformation  $S'$  on the new variables  $(w' := mw|\tau' := m\tau)$ , that is  $S'((w'|\tau')) := (w'/\tau' | -1/\tau') = (w/\tau | -1/m\tau)$ .

Thus, one obtains:

$$S \left( K_s^{(2pm+1)} \left( w \left| \frac{\tau}{m} \right. \right) \right) = S' \left( K_s^{(2pm+1)} (w'|\tau') \right) \quad (5.24)$$

and, after using the modular transformation  $S$  of  $K_s^{(2pm+1)}$  given in appendix B, the equation (5.20) is reproduced.

To prove (5.21) we have to show that the  $2 \times 2$  matrix  $A_{(m,2j)} \in \Gamma_\theta$  and the integer number  $j^* \in (1, \dots, m-1)$  exist and are unique, for any  $j$  fixed in  $1, \dots, m-1$ .

Indeed, given the matrix  $A_{(m,2j)} \in \Gamma_\theta$  its representation  $\mathcal{A}_{(m,p,2j)}$  on the characters  $K_s^{(2pm+1)}(w|\tau)$  of the free boson  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{2mp+1}$  follows by the modular transformations given in appendix B.

By definition, the generic  $2 \times 2$  matrix  $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2, \mathbb{Z})$  acts in the following way on  $(w|\tau)$ :

$$A((w|\tau)) := \left( \frac{w}{r\tau + s} \left| \frac{p\tau + q}{r\tau + s} \right. \right). \quad (5.25)$$

Thus, expanding the right hand side of (5.22) it results:

$$\left( \frac{w}{\tau} \left| \frac{-1 + 2j\tau}{m\tau} \right. \right) = \left( \frac{w}{r(\tau + 2j^*)/m + s} \left| \frac{p(\tau + 2j^*)/m + q}{r(\tau + 2j^*)/m + s} \right. \right), \quad (5.26)$$

whose solution is:

$$r = m, \quad p = 2j, \quad s = -2j^*, \quad (5.27)$$



where  $j^*$  and  $q$  have to satisfy the equation:

$$(2j)(2j^*) + qm = -1. \tag{5.28}$$

The only thing to prove now is that the integer  $j^* \in (1, \dots, m-1)$  and the **odd** integer  $q$  exist and are unique, for any fixed  $j \in (1, \dots, m-1)$ .

We observe that the equation:

$$(2j)(2j') - bm = 1 \tag{5.29}$$

has one and only one solution, with  $j'$  nonzero integer number with minimal modulo and  $b$  **odd** integer number, for any fixed  $j \in (1, \dots, m-1)$ .

Indeed, for the hypothesis  $m > 2$  prime number and  $j \in (1, \dots, m-1)$ , equation (5.29) simply expresses that  $m$  and  $2j$  are coprime numbers.

Finally, the integer  $\alpha$ , such that  $j^* = \alpha m - j' \in (1, \dots, m-1)$ , exists and is unique and, putting  $q = b - 4j\alpha$  **odd** integer, the pair  $j^*$  and  $q$  satisfy equation (5.28).

Thus, the matrix  $A_{(m,2j)}$  is:

$$A_{(m,2j)} = \begin{pmatrix} 2j & b - 4j\alpha \\ m & -2j^* \end{pmatrix}. \tag{5.30}$$

The only thing left to prove now is that the matrices  $A_{(m,2j)}$  are elements of  $\Gamma_\theta$  for any  $j \in (1, \dots, m-1)$ .

We observe that  $\det A_{(m,2j)} = -[(2j)(2j^*) + (b - 4j\alpha)m]$  is 1 using equation (5.28), so  $A_{(m,2j)} \in PSL(2, \mathbb{Z})$ . The fact that  $A_{(m,2j)} \in \Gamma_\theta$  is now a direct consequence of the characterization of  $\Gamma_\theta$  given in appendix A.

In particular, by using (A.5) this matrix can be expressed in terms of the matrices  $T^2$  and  $S$  as:

$$A_{(m,2j)} = S_{(a_1,b_1)} \times S_{(a_2,b_2)} \times \dots \times S_{(a_u,b_u)}, \tag{5.31}$$

where  $u$  is an odd positive integer,  $(a_h, b_h) \in \mathbb{Z} \times \mathbb{Z} \forall h \in (1, \dots, u)$  and  $S_{(a,b)} := T^{2a} S T^{2b}$ .  $\square$

The results of Lemma 5 make possible to give the modular transformation  $S$  for the characters  $\chi_{(\mu,f,i)}$  of the ‘‘A-P’’ twisted sector, according to:

$$\begin{aligned} & \chi_{(s,f,i)} \left( \frac{w}{\tau} \middle| -\frac{1}{\tau} \right) \\ &= \sum_{b=0}^{2pm} \sum_{e=0}^{m-1} \left( \frac{1}{m} \sum_{j=1}^{m-1} e^{2\pi i [(2j^*)(\tilde{h}_{(b,e,i)} - \frac{m}{24})]} (\mathcal{A}_{(m,p,2j)})_{(s,\mu)} e^{-2\pi i [(2j)(\tilde{h}_{(s,f,i)} - \frac{m}{24})]} \right) \\ & \cdot \chi_{(b,e,i)}(w|\tau) + \frac{1}{m} \sum_{b=0}^{2pm} \left( \frac{e^{\frac{2i\pi s b}{2mp+1}}}{\sqrt{2mp+1}} \right) \tilde{\chi}_{(b,(1,g^i))}(w|\tau). \end{aligned} \tag{5.32}$$

The previous form of the transformations  $T^2$  and  $S$  (the generators of  $\Gamma_\theta$ ) for the characters of TM shows that it is a  $\Gamma_\theta$ -RCFT, so concluding the proof of Proposition 6.  $\square$

## 6. TM as a $\Gamma_\theta$ -RCFT extension of the fully degenerate $W_{1+\infty}^{(m)}$

In all sectors of TM the residual  $c = 1$  free boson  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_{m(2mp+1)}$  can be selected out. This is well evidenced by the decomposition of the characters of TM in terms of those of  $\widehat{u(1)}_{m(2mp+1)}$ , which is the subject of the following Proposition.

**Proposition 7.** *The characters of TM have the following decomposition in terms of the characters  $K_p^{(m(2pm+1))}(w|\tau)$  of  $\widehat{u(1)}_{m(2mp+1)}$ :*

For the characters of the “P-P” untwisted sector:

$$\tilde{\chi}_b(w|\tau) = \sum_{l=0}^{m-1} \chi_l^{\widehat{su(m)}_1}(\tau) K_{(2mp+1)l+mb}^{(m(2pm+1))}(w|\tau), \quad (6.1)$$

for  $b \in \{0, \dots, 2pm\}$ .

For the characters of the “P-A” untwisted sector:

$$\tilde{\chi}_{(b,(\mathbf{1},g^i))}(w|\tau) = \frac{\eta(\tau)}{\eta(m\tau)} K_{mb}^{(m(2pm+1))}(w|\tau) \quad (6.2)$$

where  $b \in \{0, \dots, 2pm\}$  and  $i \in \{1, \dots, m-1\}$ .

For the characters of the “A-P” twisted sector:

$$\chi_{(s,f,i)}(w|\tau) = \sum_{l=0}^{m-1} N_{(l,f-2ls)}(\tau) K_{(2mp+1)l+s}^{(m(2pm+1))}(w|\tau), \quad (6.3)$$

where:

$$N_{(l,f)}(\tau) = \frac{1}{m} \sum_{j=0}^{m-1} e^{-\frac{2\pi i}{m}(2j)(\frac{f}{2}-\frac{1}{24}-\frac{l^2}{2})} \frac{\eta(\tau)}{\eta(\frac{\tau+2j}{m})}, \quad (6.4)$$

for  $s \in \{0, \dots, 2pm\}$ ,  $f \in \{0, \dots, m-1\}$  and  $i \in \{1, \dots, m-1\}$ .

*Proof.* The decomposition (6.1) is derived in Proposition 3. The decomposition (6.2) is an immediate consequence of the definitions of the characters  $\tilde{\chi}_{(b,(\mathbf{1},g^i))}(w|\tau)$ ,  $K_p^{(q)}(w|\tau)$  and of the  $\Theta$ -functions with characteristics.

Indeed:

$$\begin{aligned} K_b^{(2pm+1)}(mw|m\tau) &:= \frac{1}{\eta(m\tau)} \Theta \left[ \begin{matrix} \frac{b}{2pm+1} \\ 0 \end{matrix} \right] (m(2pm+1)w|m(2pm+1)\tau) = \\ &= \frac{\eta(\tau)}{\eta(m\tau)} \left( \frac{1}{\eta(\tau)} \Theta \left[ \begin{matrix} \frac{mb}{m(2pm+1)} \\ 0 \end{matrix} \right] (m(2pm+1)w|m(2pm+1)\tau) \right) := \frac{\eta(\tau)}{\eta(m\tau)} K_{mb}^{(m(2pm+1))}(w|\tau). \end{aligned} \quad (6.5)$$

The decomposition (6.3) follows analogously, by using the following identity, for the  $\Theta$ -functions with characteristics:

$$\Theta \left[ \begin{array}{c} \frac{\lambda}{q} \\ 0 \end{array} \right] \left( qw|q \frac{\tau + 2j}{m} \right) = \sum_{l=0}^{m-1} e^{2\pi i(2j) \frac{(ql+\lambda)^2}{2mq}} \Theta \left[ \begin{array}{c} \frac{ql+\lambda}{qm} \\ 0 \end{array} \right] (mqw|mq\tau), \quad (6.6)$$

where  $\lambda \in (0, \dots, q-1)$  and  $j \in (0, \dots, m-1)$ .  $\square$

It is worth noticing that in each sector (“P-P”, “P-A” and “A-P”) of TM the corresponding cosets with respect to  $\widehat{u(1)}_{m(2mp+1)}$  define CFTs with central charge  $c = m - 1$  whose h.w. representations can be defined in terms of those of the affine Lie algebra  $\widehat{su(m)}_1$ . In particular, the characters of the h.w. representations of these cosets are expressed in terms of those of  $\widehat{su(m)}_1$  but calculated for different specializations.

The decomposition (6.1) shows that the characters of the coset  $y$  (“P-P”-TM) /  $\widehat{u(1)}_{m(2mp+1)}$  are those of the affine Lie algebra  $\widehat{su(m)}_1$  specialized at  $\hat{\xi} = (z\rho + \tau\hat{\Lambda}_0)|_{z=0}$ .

The decomposition (6.2) and the identity (3.1) show that the characters of the coset (“P-A”-TM) /  $\widehat{u(1)}_{m(2mp+1)}$  are those of the affine Lie algebra  $\widehat{su(m)}_1$  specialized at  $\hat{\xi} = \rho/m + \tau\hat{\Lambda}_0$ .

The decomposition (6.3), (6.4) and the identity (3.30) show that the characters of the coset (“A-P”-TM) /  $\widehat{u(1)}_{m(2mp+1)}$  are written in terms of those of the affine Lie algebra  $\widehat{su(m)}_1$  specialized at  $\hat{\xi} = \rho\tau/m + \tau\hat{\Lambda}_0$  times the function  $F_{twist}^{(m)}(\tau)$ , that account for the twist with conformal dimension  $(m^2 - 1) / 24m$ .

Finally, the above observations together with the results of section 3 make possible to show:

**Proposition 8.** *The theory TM is a  $\Gamma_\theta$ -RCFT extension of the fully degenerate  $W_{1+\infty}^{(m)}$ , as it follows by the decomposition of its characters in terms of those of the fully degenerate  $W_{1+\infty}^{(m)}$ :*

*For the characters of the “P-P” untwisted sector:*

$$\tilde{\chi}_b(w|\tau) = \sum_{\mathbf{q} \in \mathbb{Z}^{(m,+)}} d_{su(m)}(\Lambda) \chi_{\mathbf{r}(b,\mathbf{q})}^{\mathbf{w}_m}(w|\tau), \quad (6.7)$$

where  $\mathbf{r}(b, \mathbf{q}) := b\mathbf{t}^T \mathbf{R}_{m,p}^{-1} + \mathbf{q}\mathbf{R}_{m,p}$ , with  $b \in \{0, \dots, 2pm\}$ .

*For the characters of the “P-A” untwisted sector:*

$$\tilde{\chi}_{(b,(1,g^i))}(w|\tau) = \sum_{\mathbf{q} \in \mathbb{Z}^{(m,+)} \cap D_m} \epsilon(w_\Lambda) \chi_{\mathbf{r}(b,\mathbf{q})}^{\mathbf{w}_m}(w|\tau), \quad (6.8)$$

where  $b \in \{0, \dots, 2pm\}$  and  $i \in \{1, \dots, m-1\}$ .

*For the characters of the “A-P” twisted sector:*

$$\chi_{(s,f,i)}(w|\tau) = F_{twist}^{(m)}(\tau) \sum_{j=0}^{m-1} \sum_{l=0}^{m-1} \sum_{a=0}^{m-1} \left\{ H_{(s,f,l,a,j)} \sum_{\mathbf{q} \in \mathbb{Z}_a^{(m,+)}} \chi_{\Lambda}^{su(m)} \left( \frac{\rho}{m}(\tau + 2j) \right) \chi_{\mathbf{r}(s,l,\mathbf{q})}^{\mathbf{w}_m}(w|\tau) \right\}, \quad (6.9)$$

where  $\mathbb{Z}_a^{(m,+)} := \left\{ \mathbf{q} \in \mathbb{Z}^{(m,+)} : \left( \sum_{i=1}^{m-1} q_i \right) = a \pmod{m} \right\}$ ,  $\mathbf{r}(s, l, \mathbf{q}) := [(s + l - a) / m] \mathbf{t}^T \times \mathbf{R}_{m,p}^{-1} + \mathbf{q} \mathbf{R}_{m,p}$  with  $\mathbf{q} \in \mathbb{Z}_a^{(m,+)}$ ,

$$H_{(s,f,l,a,j)} := \frac{1}{m} \mathbf{S}_{0,a}^{\widehat{su(m)}_1} e^{\frac{2\pi i}{m} j [2ls + l^2 - f + a(m-a)]}, \quad (6.10)$$

$s \in \{0, \dots, 2pm\}$ ,  $f \in \{0, \dots, m-1\}$  and  $i \in \{1, \dots, m-1\}$ .

In (6.7), (6.8) and (6.9)  $\Lambda$  is always defined by  $\mathbf{q}$  according to (4.35).

*Proof.* Equation (6.7) is the subject of Proposition 4. The proof of (6.8) is a consequence of decomposition (6.2) and of the identities (3.1) and (3.2). Indeed, the same considerations of Proposition 4 imply:

$$\tilde{\chi}_{(b,(\mathbf{1},g^i))}(w|\tau) = \sum_{\mathbf{q} \in \mathbb{Z}^{(m,+)}} \chi_{\Lambda}^{su(m)} \left( \frac{\rho}{m} \right) \chi_{\mathbf{r}(b,\mathbf{q})}^{\mathbf{w}_m}(w|\tau), \quad (6.11)$$

that leads to (6.8) by (3.4).

Finally, the proof of (6.9) follows by the decomposition (6.2) and the Corollary 1. Indeed, this last one implies:

$$\frac{\eta(\tau)}{\eta\left(\frac{\tau+2j}{m}\right)} = F_{twist}^{(m)}(\tau) \sum_{a=0}^{m-1} e^{2\pi i(2j)\left(h_{\widehat{\Lambda}_a} - \frac{1}{24m}\right)} \mathbf{S}_{0,a}^{\widehat{su(m)}_1} \chi_{\widehat{\Lambda}_a}^{\widehat{su(m)}_1} \left( \frac{\rho}{m}(\tau+2j) \mid \tau \right), \quad (6.12)$$

that becomes by (2.7):

$$\frac{\eta(\tau)}{\eta\left(\frac{\tau+2j}{m}\right)} = F_{twist}^{(m)}(\tau) \sum_{a=0}^{m-1} e^{2\pi i(2j)\left(h_{\widehat{\Lambda}_a} - \frac{1}{24m}\right)} \mathbf{S}_{0,a}^{\widehat{su(m)}_1} \sum_{\Lambda \in P_+ \cap \Omega_a} \chi_{\Lambda}^{su(m)} \left( \frac{\rho}{m}(\tau+2j) \right) \chi_{\Lambda}^{\mathcal{W}_m}(\tau). \quad (6.13)$$

Now, following the same consideration developed in the proof of Proposition 4 and using the decomposition (6.3) it results:

$$\begin{aligned} \chi_{(s,f,i)}(w|\tau) &= F_{twist}^{(m)}(\tau) \sum_{j=0}^{m-1} \sum_{l=0}^{m-1} \sum_{a=0}^{m-1} \\ &\left\{ \frac{1}{m} \mathbf{S}_{0,a}^{\widehat{su(m)}_1} e^{2\pi i(2j)\left(\frac{((2pm+1)l+s)^2}{2m(2pm+1)} + h_{\widehat{\Lambda}_a} + \frac{m^2-1}{24m} - h_{(s,f,i)}\right)} \right. \\ &\left. \sum_{\mathbf{q} \in \mathbb{Z}_a^{(m,+)}} \chi_{\Lambda}^{su(m)} \left( \frac{\rho}{m}(\tau+2j) \right) \chi_{\mathbf{r}(s,l,\mathbf{q})}^{\mathbf{w}_m}(w|\tau) \right\}, \end{aligned} \quad (6.14)$$

where  $\mathbf{r}(s, l, \mathbf{q}) := [(s + l - a) / m] \mathbf{t}^T \mathbf{R}_{m,p}^{-1} + \mathbf{q} \mathbf{R}_{m,p}$  with  $\mathbf{q} \in \mathbb{Z}_a^{(m,+)}$ . Equation (6.14) coincides with (6.9) taking into account that:

$$h_{\widehat{\Lambda}_a} := \frac{(\widehat{\Lambda}_a, \widehat{\Lambda}_a + 2\rho)}{2(m+1)} = \frac{a(m-a)}{2m}. \quad (6.15)$$

□

## 7. Conclusions

In this paper, we found an RCFT extension of the fully degenerate  $W_{1+\infty}^{(m)}$  chiral algebra. The relevance of the last chiral algebra for the description of the Quantum Hall Fluid plateaux has been underlined in [22]. The TM model has been applied to the description of such a phenomenon in [42, 1] and to other physical systems in [44]. An interesting property of such an RCFT is the possibility of defining different extensions of  $W_{1+\infty}^{(m)}$  in any sector of the orbifold. That relies deeply on the different multiplicities of the physical vectors appearing in the spectrum of each sector. Moreover, we found that there is a one to one correspondence between the CFTs with chiral symmetry  $su(m) \otimes W_{1+\infty}^{(m)}$  [19] and the so called minimal models [22]. They are simply two different sectors of TM, which so gives a consistent RCFT containing fully degenerate representations of  $W_{1+\infty}^{(m)}$  and satisfying the modular invariance constraint (i.e. it is a completion of the minimal model given in [22]).

## Acknowledgments

G.N. was supported by a postdoctoral fellowship of the Ministère français délégué à l'Enseignement supérieur et à la Recherche.

## A. The $\Gamma_\theta$ group

The group  $\Gamma_\theta$  denotes, according to the definition given in section 13.4 of the Kac's book [27], the subgroup of the modular group  $PSL(2, \mathbb{Z})$  generated by  $T^2$  and  $S$ . A matrix representation of the generators  $T^2$  and  $S$  is:

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

$\Gamma_\theta$  is the group of elements:

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) : a + d, b + c \text{ even}, a + b \text{ odd} \right\}. \quad (\text{A.2})$$

Any element  $A$  of  $\Gamma_\theta$  can be represented as follows:

$$A = \begin{cases} T^{2a} & \forall a \in \mathbb{Z} \\ S_{(a_1, b_1)} \times S_{(a_2, b_2)} \times \cdots \times S_{(a_r, b_r)} & \forall (a_j, b_j) \in \mathbb{Z} \times \mathbb{Z}, \forall j \in (1, \dots, r) \\ & \forall r \in \mathbb{N} \end{cases} \quad (\text{A.3})$$

where  $S_{(a,b)} = T^{2a} S T^{2b}$ . Thus, the characterization given for the subgroup  $\Gamma_\theta$  is a direct consequence of the form of these matrices; indeed:

$$T^{2a} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \quad (\text{A.4})$$

and

$$\prod_{j=1}^r S_{(a_j, b_j)} = \begin{cases} \begin{pmatrix} 2\alpha + 1 & 2\beta \\ 2\gamma & 2\delta + 1 \end{pmatrix} & \text{for } r \text{ even} \\ \begin{pmatrix} 2\alpha & 2\beta + 1 \\ 2\gamma + 1 & 2\delta \end{pmatrix} & \text{for } r \text{ odd} \end{cases}, \quad (\text{A.5})$$

where  $\alpha, \beta, \gamma, \delta$  are integers and depend on  $(a_j, b_j)$  and  $r$ .

### B. The $\Gamma_\theta$ -RCFT $\widehat{u(1)}_p$

Let us recall that the quantized free boson field has the following expansion:

$$\varphi(z, \bar{z}) := \varphi_0 + \phi(z) + \bar{\phi}(\bar{z}), \quad (\text{B.1})$$

where:

$$\begin{cases} \phi(z) := ia_0 \ln(1/z) + i \sum_{k \in \mathbb{Z} - \{0\}} a_k z^{-k}/k \\ \bar{\phi}(\bar{z}) := i\bar{a}_0 \ln(1/\bar{z}) + i \sum_{k \in \mathbb{Z} - \{0\}} \bar{a}_k \bar{z}^{-k}/k \end{cases}, \quad (\text{B.2})$$

$\{a_k\}_{k \in \mathbb{Z}}$  and  $\{\bar{a}_k\}_{k \in \mathbb{Z}}$  are two independent chiral Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$  and the zero-mode  $\varphi_0$  is a conjugate operator to  $a_0$  ( $\bar{a}_0$ ):

$$[a_n, a_m] = n\delta_{n,m}, \quad [a_n, \bar{a}_m] = 0, \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n,m}, \quad [\varphi_0, a_m] = i\delta_{0,m}, \quad [\varphi_0, \bar{a}_m] = i\delta_{0,m}. \quad (\text{B.3})$$

The free boson  $\widehat{u(1)}$  with chiral algebra  $\mathfrak{A}(\widehat{u(1)})$ , generated by the modes of the conserved current  $i\partial\phi(z)$ , is a chiral CFT with stress energy tensor  $T(z) := (-1/2):\partial\phi(z)\partial\phi(z):$  and central charge  $c = 1$ . This CFT has a one parameter family of h.w. vectors:

$$a_0 |\alpha\rangle = \alpha |\alpha\rangle, \quad a_n |\alpha\rangle = 0 \quad \text{for } n > 0, \quad (\text{B.4})$$

with corresponding (h.w.) irreducible positive energy module  $H_\alpha := \{a_{-n_q}^{m_q} \cdots a_{-n_1}^{m_1} |\alpha\rangle$  with  $n_i > 0, m_i > 0, q > 0\}$ . The module  $H_\alpha$  is the irreducible module of the Virasoro algebra with h.w.  $\alpha$  and conformal dimension<sup>10</sup>  $\alpha^2/2$ , as the expansions:

$$L_n := \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} a_m \quad \forall n \in \mathbb{Z} - \{0\}, \quad L_0 := \sum_{m \in \mathbb{Z}} a_{-m} a_m + \frac{1}{2} a_0^2 \quad (\text{B.5})$$

in the modes of the Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$  imply.

The free boson  $\widehat{u(1)}$  of course is not a rational CFT. RCFT extensions [45] of it are defined compactifying the free boson field on a circle of rational square radius and correspondingly introducing an extension of the Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$ . More explicitly, the compactification condition on the circle with radius  $r = \sqrt{2p}$ ,  $p$  positive integer, is:

$$\varphi(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = \varphi(z, \bar{z}) + 2\pi r m, \quad (\text{B.6})$$

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<sup>10</sup>It is, in fact, the lowest eigenvalue of  $L_0$  in the module  $H_\alpha$ .

where  $m \in \mathbb{Z}$  is called winding number [29]. The compactification condition has the only effect to influence the zero-mode  $a_0$  ( $\bar{a}_0$ ) of the free boson field. In particular, to obtain well defined vertex operators under the compactification condition, the possible eigenvalues of  $a_0$  are restricted to the following values  $\alpha_n := n/r$ ,  $n \in \mathbb{Z}$ . So, the h.w. vectors of the compactified free boson  $\widehat{u(1)}$  get reduced to:

$$a_0 |\alpha_n\rangle = \alpha_n |\alpha_n\rangle, \quad a_r |\alpha_n\rangle = 0 \quad \text{for } r > 0, \quad L_0 |\alpha_n\rangle = h_n |\alpha_n\rangle, \quad (\text{B.7})$$

where  $h_n := \alpha_n^2/2$ . The irreducible module corresponding to the h.w. vector  $|\alpha_n\rangle$  is denoted by  $H_n$  and the corresponding character is:

$$\text{Tr}_{H_n} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \frac{1}{\eta(\tau)} q^{h_n} e^{2\pi i w r \alpha_n}, \quad (\text{B.8})$$

where  $J := r a_0$  is the conformal charge.

The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{2p})$  extension of the Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$  is defined by adding to it the modes of the two chiral currents  $\Gamma_{2p}^\pm(z) := :e^{\pm i\sqrt{2p}\phi(z)}:$ .  $\Gamma_{2p}^\pm(z)$  are uniquely characterized as the vertex operators with lowest nonzero conformal dimension satisfying the requirements of well definition with respect to the compactification condition (B.6) (i.e. they have to be invariant under  $\phi \rightarrow \phi + 2\pi r$ ) and of locality (i.e. integer conformal dimension). The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{2p})$  has  $r^2 = 2p$  h.w. vectors  $|\alpha_l\rangle$ , those with  $\alpha_l = l/r$  for  $l \in \{0, \dots, 2p-1\}$ . The corresponding irreducible modules are  $H_l^{(2p)} := \bigoplus_{u \in \mathbb{Z}} H_{l+u(2p)}$  and so the corresponding characters are:

$$K_l^{(2p)}(w|\tau) := \text{Tr}_{H_l^{(2p)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \frac{1}{\eta(\tau)} \sum_{u \in \mathbb{Z}} q^{p(\frac{l}{2p} + u)^2} e^{2\pi i w (2p)(\frac{l}{2p} + u)}, \quad (\text{B.9})$$

for  $l \in \{0, \dots, 2p-1\}$ , which in terms of  $\Theta$ -functions with characteristics are written as:

$$K_l^{(2p)}(w|\tau) = \frac{1}{\eta(\tau)} \Theta \left[ \begin{matrix} \frac{l}{2p} \\ 0 \end{matrix} \right] (2pw|2p\tau). \quad (\text{B.10})$$

The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_{2p})$  defines an RCFT because its characters  $K_l^{(2p)}(w|\tau)$  define a  $2p$ -dimensional representation of the entire modular group  $PSL(2, \mathbb{Z})$ . That immediately follows by the modular transformations of the characters  $K_l^{(2p)}(w|\tau)$ :

$$K_l^{(2p)}(w|\tau+1) = e^{i2\pi(\frac{l^2}{4p} - \frac{1}{24})} K_l^{(2p)}(w|\tau), \quad K_l^{(2p)}\left(\frac{w}{\tau} \middle| -\frac{1}{\tau}\right) = \frac{1}{\sqrt{2p}} \sum_{l'=0}^{2p-1} e^{\frac{2i\pi l' l}{2p}} K_{l'}^{(2p)}(w|\tau). \quad (\text{B.11})$$

We denote this RCFT simply with  $\widehat{u(1)}_{2p}$ .

Here, we want to define a class of  $\Gamma_\theta$ -RCFT extensions of the Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$ . It can be done by admitting for the free boson CFT a compactification condition with odd square radius  $r^2 = p$ ,  $p$  odd.

The same analysis as above holds with the only difference that the chiral algebra  $\mathfrak{A}(\widehat{u(1)}_p)$  giving the extension of the Heisenberg algebra  $\mathfrak{A}(\widehat{u(1)})$  is now defined by adding

to it the modes of the two chiral currents  $\Gamma_p^\pm(z) := :e^{\pm i\sqrt{p}\phi(z)}:$ , which are now locally anticommuting Fermi fields with half integer ( $p/2$ ) conformal dimensions.

The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_p)$  has  $r^2 = p$  h.w. vectors  $|\alpha_l\rangle$ , those with  $\alpha_l = l/r$  for  $l \in \{0, \dots, p-1\}$ . The corresponding irreducible modules are  $H_l^{(p)} := \bigoplus_{u \in \mathbb{Z}} H_{l+up}$  and so the corresponding characters are:

$$K_l^{(p)}(w|\tau) := Tr_{H_l^{(p)}} \left( q^{(L_0 - \frac{m}{24})} e^{2\pi i w J} \right) = \frac{1}{\eta(\tau)} \sum_{u \in \mathbb{Z}} q^{\frac{p}{2}(\frac{l}{p} + u)^2} e^{2\pi i w p(\frac{l}{p} + u)}, \quad (\text{B.12})$$

where  $J := ra_0$ ,  $l \in \{0, \dots, p-1\}$ , or in terms of  $\Theta$ -functions:

$$K_l^{(p)}(w|\tau) = \frac{1}{\eta(\tau)} \Theta \left[ \begin{matrix} \frac{l}{p} \\ 0 \end{matrix} \right] (pw|p\tau). \quad (\text{B.13})$$

The chiral algebra  $\mathfrak{A}(\widehat{u(1)}_p)$  defines a  $\Gamma_\theta$ -RCFT because its characters  $K_l^{(p)}(w|\tau)$  define a  $p$ -dimensional representation of the modular subgroup  $\Gamma_\theta$ . Indeed, the modular transformation of  $K_l^{(p)}(w|\tau)$  are:

$$K_l^{(p)}(w|\tau + 2) = e^{i4\pi(\frac{l^2}{2p} - \frac{1}{24})} K_l^{(p)}(w|\tau), \quad K_l^{(p)}\left(\frac{w}{\tau} \middle| -\frac{1}{\tau}\right) = \frac{1}{\sqrt{p}} \sum_{l'=0}^{p-1} e^{\frac{2i\pi l'l}{p}} K_{l'}^{(p)}(w|\tau). \quad (\text{B.14})$$

We denote this  $\Gamma_\theta$ -RCFT simply as  $\widehat{u(1)}_p$ ,  $p$  odd.

Furthermore, we observe that the  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_p$ ,  $p$  odd, coincides with the  $\Gamma_\theta$ -projection of the ordinary RCFT  $\widehat{u(1)}_{4p}$ . This is an immediate consequence of the relations among the corresponding characters:

$$K_l^{(p)}(w|\tau) = K_{2l}^{(4p)}(w|\tau) + K_{2(p+l)}^{(4p)}(w|\tau), \quad (\text{B.15})$$

for  $l \in \{0, \dots, p-1\}$ . Finally, the operator content of  $\Gamma_\theta$ -RCFT  $\widehat{u(1)}_p$  does not coincide with that of the ordinary RCFT  $\widehat{u(1)}_{4q}$ . Indeed, the h.w. representations corresponding to  $a/\sqrt{4q}$  and  $(a+2q)/\sqrt{4q}$ , for  $a \in \{0, \dots, q-1\}$ , of  $\widehat{u(1)}_{4q}$  do not belong to  $\widehat{u(1)}_q$ .

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